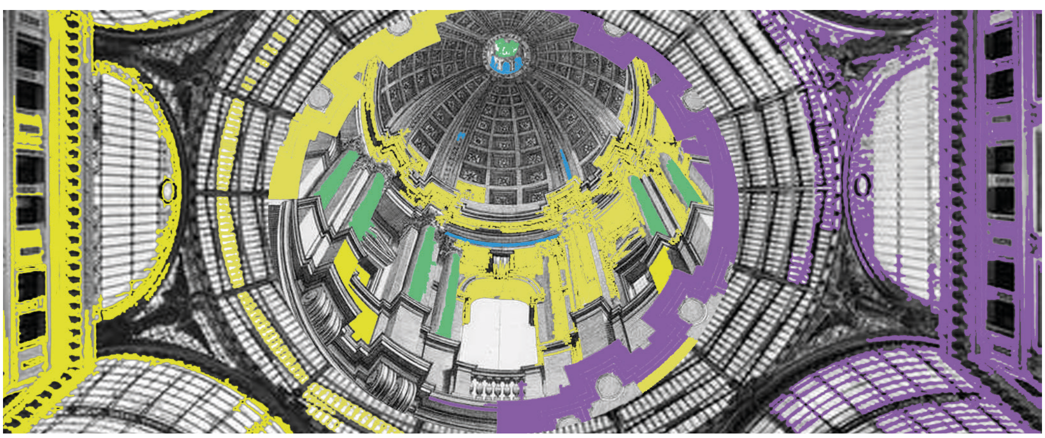
The Study of the Past for the Overcoming of the Future. The Study of the Sphere in the Science of Representation

Laura Inzerillo

Abstract

The geometric construction of complex figures and related elements often presents difficulties that are not always easily overcome, even with the use of cutting-edge software. Knowledge of simplified geometric procedures provides the necessary tools for potential implementations of digital construction algorithms. The geometric construction in orthogonal axonometry of the sphere, based on construction procedures founded on the homology of reflection [Schreiber 2002 p. 39; Inzerillo 2012, p. 5], ensures a rigorous yet significantly simplified representation. The study presented, however, demonstrates virtuosity on the theme, such as circles parallel to each other and orthogonal to the equatorial plane, tangent cones, and distances of points from the spherical surface [Mileikovskyi 2019, p. 100].

Keywords
Descriptive geometry, CAD, oblique axonometry, orthogonal axonometry, geometric methodology.



Graphic overlapping of portions of different domes and from different historical periods with the related clouds detected by the author: the study of the past for the overcoming of the future.

doi.org/ 10.3280/oa-1430-c953

“Change is the law of life, and those who look only to the past or present are sure not to see the future”.
J. F. Kennedy, 1962 [Ratcliffe 2018, p. 107]

Introduction

It is interesting to note how the science of representation may sometimes seem out of step with the times, but it is a constantly evolving field. Although some complex geometries present challenges, especially regarding their representation, there is always new research and studies that can offer solutions. This study, for example, focuses on simplifying approaches to represent the sphere in orthogonal axonometry, using meridians, parallels, and spherical caps [Brezov, Kancheva, Nikova 2013, p. 200]. It is a great way to tackle difficulties and make complex concepts more accessible!

The sphere

The properties of the sphere are numerous, and equally important are the differences and affinities that relate it to other quadrics; these are surfaces representable by second-degree equations. They are distinguished into elliptical point quadrics, such as the sphere, ellipsoid, elliptical paraboloid, hyperboloid of two sheets [Koch 1998, p. 97]; parabolic point quadrics, such as cylinders and cones, which have conical plane sections; and hyperbolic point quadrics, such as hyperbolic paraboloids and hyperbolic hyperboloid [Hirsch 2002, p. 123], depending on their relationship with infinity.
All points on the sphere are equidistant from its center by a measure known as the radius. The plane section of the sphere is always and only a circle, which is maximized if the cutting plane contains the center of the sphere, which is also the center of the circular section; further away from the center the cutting plane is, the smaller its radius becomes. When it is at a distance equal to the radius, the section reduces to a point and the plane becomes tangent, orthogonal to the radius that contains the point of tangency. A cylinder that projects a sphere is a right circular cylinder [Sugihara 2002, p. 75]; it tangents according to a maximum circle whose plane contains the center and is orthogonal to the axis of the cylinder. The radii projecting onto plane π, along with points on the sphere common to the tangent circle of the cylinder, defines a closed conic on π. For any projecting direction, it is an ellipse, except when the projection is orthogonal to π as in orthogonal axonometry; in this case, the conic intersection and the tangent circle are parallel and therefore equal. Thus, in orthogonal axonometry, the projection of a sphere σ is always and only a circle, whose center *O′* is the projection of the center of σ and whose radius is equal to that of σ. No radius projecting a point of σ can be outside the tangent cylinder; no projection point can fall outside it, which is therefore the apparent contour *cap′* of the projections of points on *σ*.
With reference to the triplet *x′y′z′* (fig. 1), the center *O′* must always be associated with a second projection, for example, that onto *xy*, *0xy′*, which may be lower or higher than *O′*; if I choose the center of σ belonging to *xy*, there is uniqueness for both *O′* and *0xy′*.
The projection of the maximum circle belonging to *xy* is an ellipse *eq′* with center *0xy′*; the major axis *1xy′-0xy′-2xy′* is parallel to *txy*; the minor semi-axis *0xy′-4xy′* is the shortening of the inverted radius, at maximum slope of *xy* relative to π, *0xy′-3*, parallel to *H-O** for projection along *O-O′*; the horizontal circle, by convention, is called the equator; I denote it as eq′. Among diameters, an important role is played by that orthogonal to the maximum horizontal section, called the polar axis, whose ends are referred to as north pole and south pole in projection *N′* and *S′*; the polar radius is parallel to *z* and shortens according to *z* on *z′*; the radius *0xy′-N** orthogonal to *0xy′-3* projects onto the vertical line drawn through *0xy′*, yielding *N′*; *S′* is its symmetric point with respect to *0xy′*.
Draw its projection; I want to give its center *0′* a positive elevation relative to *xy*, moving *0xy′* downwards; with respect to the adopted reference system, I have produced a double movement of σ; I have raised it and brought it closer; the projection seems unchanged, motionless, and yet it is the projection of another *σ*, or rather the same one but in a different

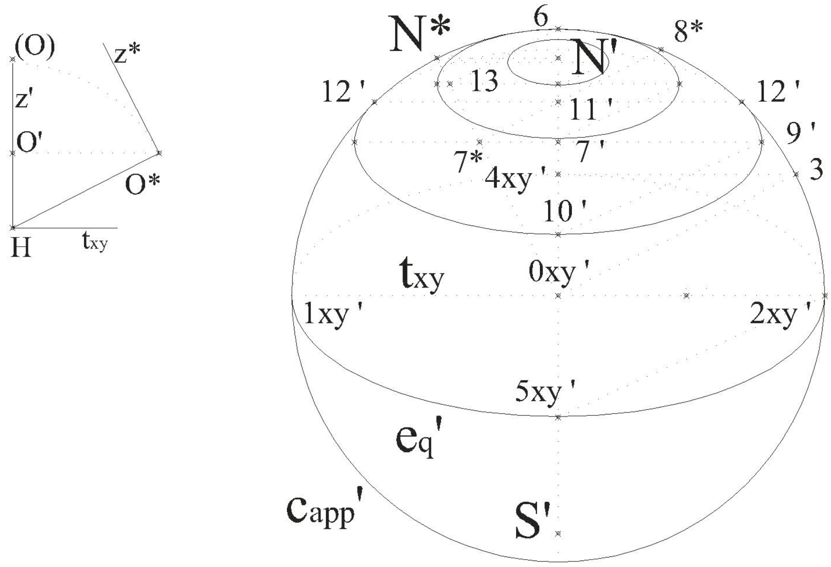


Fig. 1. Homological relationship and representation of the sphere and its parallels (image by the author).

position; if the sphere is resting on *xy, 0xy′* is superimposed on *S′*. I insist on considering the distance *0′-0xy′*; define *σ₁′* with *01′-01xy′* and *σ₂′* with *02′-02xy′*, if I move *02xy′* relative to *01xy′*, while keeping the image of the apparent contours intact, I can completely alter the positional arrangement of high and near between the two represented elements, regardless of the reference. Therefore, for the visual perception of reality, the graphic representation of the projection *0xy* of the centers is essential; their absence, due to forgetfulness, underestimation, or ignorance of the role of *0xy* blocks the reading and understanding of the structure of the image; for example, computer-generated images are often misleading. I note in figure *0xy′* on horizontal *txy*, a trace of plane *xy* on *π*, having chosen for *eq′* membership in *xy*; this facilitates the application of the defined flipping homology given by an external homological ratio; it is a convenient freedom without contraindications, like having chosen to superimpose *0′* on *0xy′*. Depending on the adopted homological ratio *H-O′:H-(O), 4xy′* can move away from *0xy′*, while simultaneously *N′* approaches *0xy′*; *N′* position is easily predictable; the ratio may not be expressed separately but decided by assigning *0xy′-4xy′;* then mark *4xy′-3, 3-0xy′*, orthogonal *0xy′-N***, and *N*-N′*.
I can decide that *N′* coincides with 6, meaning that the polar axis is parallel to *π* and the equatorial plane is orthogonal to *π*; in this case, the equator will be represented solely by the diameter *1′-0′-2′* and not as an ellipse; this is an orthogonal frontal projection of σ and not an axonometry; therefore, *N′* cannot be confused with 6; this consideration is very serious and important, as this 'oversight' is not rare. The visualization of *σ* is realized by *capp′*, *eq′*, and *N′-0xy′-S′*; it can be enriched by the representation of meridians, maximum sections on planes orthogonal to the equatorial plane conducted through *N′ -0xy′-S′*, minor planar sections parallel to the equator, called parallels, minor planar sections parallel to the meridians, generic planar sections, maximum or minor. There can be a coincidence of multiple planar sections, resulting in zones, caps, areas, triangles, wedges, sails, spherical grids circumscribed to regular polyhedra, semi-regular ones, etc., with simple or composite elements, aggregated or exploded; the minimum arc that separates two points on the spherical surface belongs to the circumference that contains both points and the center of the *σ*. All of this can be thought of in relation to a σ that presents a non-vertical polar axis; additionally, I must delve into issues regarding point membership to σ, distance from points to *σ*, intersection of lines with *σ*, tangent planes, distance from planes to *σ*, exposure of *σ* to a point, visualization of *Q* from *P* allowed or hindered by *σ*, determination regarding *P* of space visible or hidden by *σ* under static or dynamic conditions, particular paths of a point on *σ* etc.

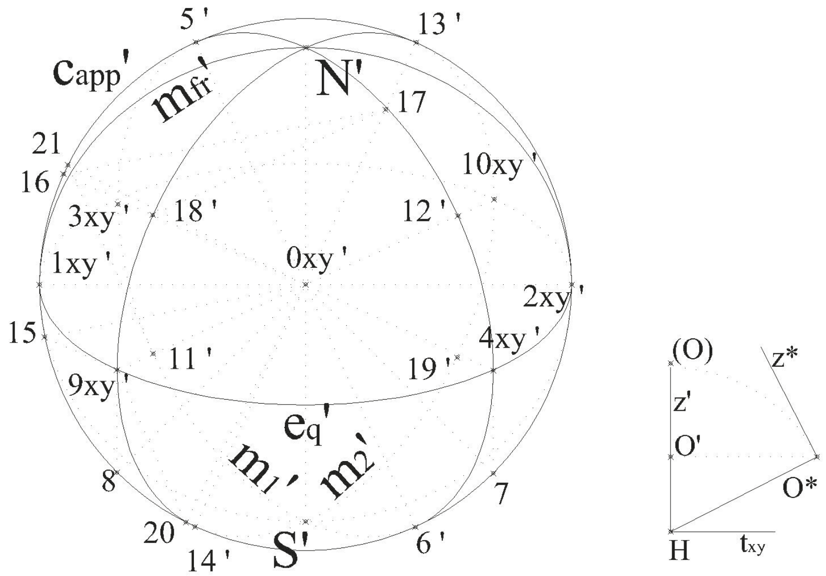


Fig. 2. Representation of the sphere and its meridians, once the homological relationship has been defined (image by the author).

Let's focus on the drawing of the parallels, circular plane sections parallel to the equatorial plane; on the upper polar semi-axis inverted, *0xy'-N**; at a distance of *0xy'-7** from the center of the sphere, the radius of the section is *7*-8**; as the distance increases, *7** approaches *N**, and the radius *7*-8** decreases; just as *N** projects onto *N'*, so does *7** project onto *7'*, and *7'-9'=7*-8** is the semi-major axis, while *9'-10'*, parallel to *2xy'-5xy'*, determines the semi-minor axis *7'-10'*; I can construct the ellipse projection of the parallel, having derived its axes. The line *7*-8** intersects *0xy'-N'* at point *11'*; the line parallel to the major axis drawn through point *11'* identifies on the apparent contour capp' points *12'* and *12'* of tangency; moving point *7** to point *13*, which refers back to point *8*, falls on point *6*; point *6* becomes the unique contact point of the ellipse relative to the parallel, whose distance from the center is equal to *0xy'-13*. The ellipse *eq'* is half hidden, the ellipse of the first parallel is hidden for less than half, and the ellipse of the second parallel is the first to be fully visible; if point 7 continues to approach *N**, then the ellipse of that parallel appears fully visible. *N'* always remains included in every fully visible parallel; the extreme parallel at *N'* has a null radius; all parallels that are fully visible, starting from that tangent at point *6*, show a densification between points *6* and *N'* of one end of the minor axis relative to the other.
Let's remember that *z'* is on the sheet in a vertical direction, but the spatial axis *z*, relative to us as observers, is inclined towards us; *z* is vertical in the sense that it is orthogonal to *xy*, which is not horizontal but also inclined relative to us. Let's look at the points of a parallel; they are all at the same altitude relative to *xy*; the 10th is not lower than the 9th, and the 6th is not higher than *N'*; the reality communicated by the image is the intelligent reading of appearance. If we erase the lower semicircle, we isolate a spherical cap; it is still a cap, but not hemispherical if we stop the image at a parallel above or below the equator; we can consider a concentric hemisphere with a smaller radius that creates a hollow volume with constant thickness; let's isolate the lower hemisphere; we obtain a hemispherical cup, with the thickness of the edge fully visible; the two ellipses of the edge are homothetic to each other. I make similar considerations for an axonometric view from below and not from above; the hemisphere shows eq' in its entirety, the upper half of *capp'*; from the first parallel, only the arc of the 10th allowed by *eq'* remains visible; the rest is hidden; however, in an urban axonometric view, meaning with a homological ratio close to one, the ellipse *eq'* approximates to a circle and reveals point *N'*, which will appear close to *0xy'*; then nearby parallels to *N* will also be visible, those without contact with *capp'*, as permitted by *eq'*.

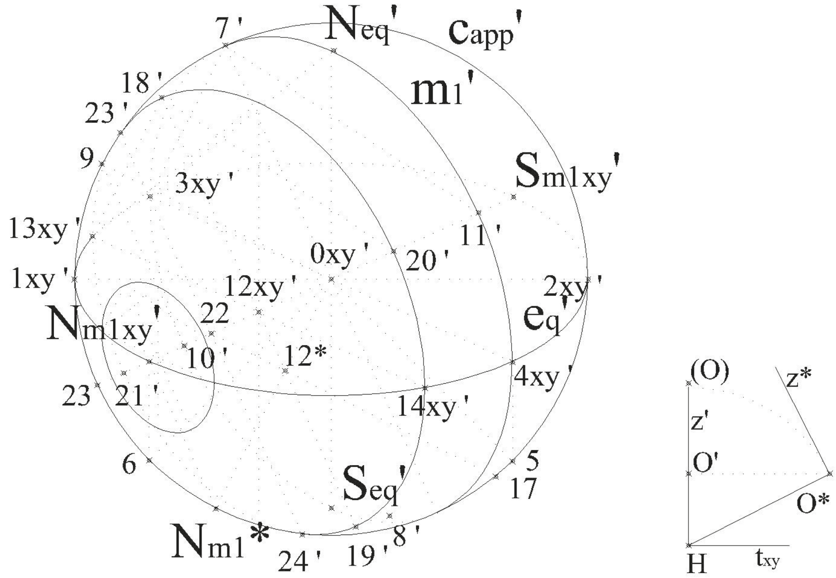


Fig. 3. Circles parallel to a chosen meridian (image by the author).

We can consider the representation of the figure 2. To derive the construction of the meridian with orientation *9’xy-0’xy-10’xy* and conjugate diameter *N’S’*, it is necessary to find the trace of the plane that contains the two aforementioned diameters. If we assume that *9’xy-0’xy-10’xy* is, for example, the *x*-axis, and the diameter *N’S’* is the *z*-axis, we know that the trace of the *xz* plane is orthogonal to the *y’*-axis, a projection of *y* orthogonal to *x*. Therefore, we must find *y’*, a projection of *y* orthogonal to *x*, then we flip *x’* onto *x* in the overturning circle that coincides with the apparent contour circle, i.e., *O’xy-8*, overturning of *O’xy-9’xy*. We consider *O’xy-7* orthogonal to *O’xy-8* (*y* overturned), whose corresponding value is *O’xy- 4’xy* (*y’*). At this point we can consider the trace of the *xz* plane for *O’xy* which intercepts the points *13’* and *14’* on the apparent contour.
The minor axis *11’-0xy’-12’* is obtained from *N’* for the orthogonal affine homology with axis *5’-0xy’-6’*. The diameter *9xy’-0xy’-10xy’* defines the meridian *m2’*, orthogonal to the previous one; its major axis *13’-0xy’-14’* is orthogonal to *3xy’-0xy’-4xy’*; the minor axis *18’-0xy’-19’* is derived from *9xy’* for the orthogonal affine homology with axis *13’-0xy’-14’*; I conduct the line *9xy’-15*, orthogonal to *13’-0xy’-14’*, then line *15-0xy’*, line *0xy’-16*, line *16-17* parallel to line *15-0xy’*, and finally homologous line 17-18’ parallel to line 9xy’-0xy’, and finally line 18’-0xy’.
I observe that *9xy’-0xy’-10xy’*, with respect to the meridian *m1’*, has the same function as the polar axis with respect to the equator; if I draw through *9xy’* the line *9xy’-20*, parallel to the axis *5’-0xy’-6’*, the line *20-0xy’*, the orthogonal line *0xy’-21*, and the parallel to *5’-0xy’-6’* drawn through point *21* retrieves the endpoint *11’* of the minor axis of *m1’*; for accuracy, I should denote with *Neq’* the pole *N’* relative to the equator, with *Nm1’* the pole *9xy’* relative to *m1’*, and with *Nm2’* the pole *4xy’* relative to *m2’*. The frontal meridian *mfr’* is defined by *N’-0xy’-S’*, major axis of the equator and by *N’-0xy’-S’*, polar axis and minor axis; the polar axis of capp’ is point *0xy’*, and in this case *capp’* acts as an equator and every diameter represents one of its meridians; the corresponding parallels are circular and concentric; parallels and meridians of *capp’* used as an equator cannot be interpreted except through the projection of their projection on *xy*, being parallel and orthogonal to π.
I consider a meridian, its corresponding polar axis, and some minor parallel sections; in the figure 3, I report *capp’*, *eq’*, *m1’* defined by *3xy’-0xy’-4xy’*, trace of the meridian on the equatorial plane and conjugate diameter to *Neq’-0xy’-Seq’*, polar axis relative to eq’; following the known path, from *4xy’-5-0xy’-6-Nm1xy’* I identify *N'm1'xy-0xy’-S'm1'xy*, diameter of *eq’* conjugate to *3xy’-0xy’-4xy’* and polar axis relative to *m1’*.
On *Nm1-0xy’*, inverted from the polar radius, I choose the distance *0xy’-12*, which projects onto *Nm1’-0xy’* at *12xy’*, the center of the constructing parallel; having drawn *eq’*, I identify the chord *13xy’-12xy’-14xy’*, which is the diameter that defines the ellipse along

with the vertical conjugate *15′-12xy′-16′* obtained by drawing a parallel through *13xy′ to 3xy′-Neq′*; the major axis *18′-12xy′-19′* is parallel to *7′-0xy′-8′*; the semi-major axis is equal to *12-17*, orthogonal to *Nm1-0xy′*; the semi-minor axis *12xy′-20′* is obtained by drawing a parallel through *18′* to *7′-11′*; line *17′-12xy′* intersects the polar axis *N′m1′xy-0xy′-S′m1′xy* at point *22*; the orthogonal to *N′m1′xy-0xy′-S′m1′xy* drawn through point *22* identifies on *capp′* the points of contact, *23′, 24′*. Moving point *12** towards *Nm1*, point *22* shifts towards *Nm1xy′*; when point *22* reaches extreme point *23*, the ellipse has only one point of tangency and is the first to be fully visible; its inverted center will be the intersection between *Nm1-0xy′* and the orthogonal to *Nm1-0xy′* drawn through point *23*; the projection of this point, inverted center, onto *N′m1xy′-0xy′* is the projection of the center of the ellipse; the semi-major axis is equal to the segment of orthogonal that goes from the inverted center to circumference *capp′*; when segment *12*-22* in its movement surpasses point *23*, the radius approaches zero value which occurs when point *12** overlaps with *Nm1**; in this mentioned vicinity, the ellipse circumscribes *Nm1′* and has no contacts, and all parallels are entirely visible.
We consider figure 4. A cone with vertex *V(V′, Vxy′)* is tangent to a sphere *σ*, of which the following are known: *0xy′*, *capp′*, *eq′*, *N′-0xy′-S′*, according to a minor circular section of the maximum and with a horizontal position α orthogonal to *V-0xy*; I consider the vertical projecting plane for *V-0xy*, with trace *Vxy′-0xy′*, intersecting σ according to a meridian section m′ defined by the diameter *3xy′-0xy′-4xy′*, common to eq′ and the vertical conjugate *N′-0xy′-S′*, polar axis relative to the equator; I determine the diameter *7xy′-0xy′-8xy′* conjugate to *3xy′-0xy′-4xy′*; it is already known that, in the ellipse m′, the line *9′-0xy′-10′*, orthogonal to *7xy′-0xy′-8xy′*, is the major axis and that the minor axis *13′-0xy′-14′*, superimposed on *7xy′-0xy′-8xy′*, is obtained along the path *7xy′-11-0xy′-12-13′*. The points *V′-15′* and *V′-16′,* tangent from *V′* to *m′*, define *15′-17′-16′*, the diameter of the contact section *ct′* between *σ* and the cone; the conjugate is parallel to *7xy′-0xy′-8xy′* drawn through point *17*, intersection between *V′-0xy′* and *15′-16′* and midpoint of the polar chord. To determine the real distance *17′-0xy′*, or rather the true dimensions of the legs *0xy′-17xy′* and *17xy′-17′*, I consider *0xy′-18* as homologous to *0xy′-17xy′* and *17xy′-19* parallel to *z**, which measures *17xy′-17′;* the hypotenuse *20-21* of legs *0xy′-18* and *17xy′-19*, reported separately, is compared at point *20-22* with line segment *20-23*, radius of capp′, orthogonal to *7xy′-0xy′-8xy′;* semicord *22-24*, radius of the tangent to *σ*.
The line *tαxy′* parallel to *7xy′-0xy′-8xy′* conducted through the point *30*, intersection between *16′-17′-15′* and *0xy′-17xy′-Vxy′*, is the trace on the equatorial plane of the plane alpha of the conic section *ct,* and it is the axis of homology with center *V′* between the *ct* and the ellipse *cxy′*, projection of the *ct* from *V* onto the equatorial plane. From *cxy′,* not

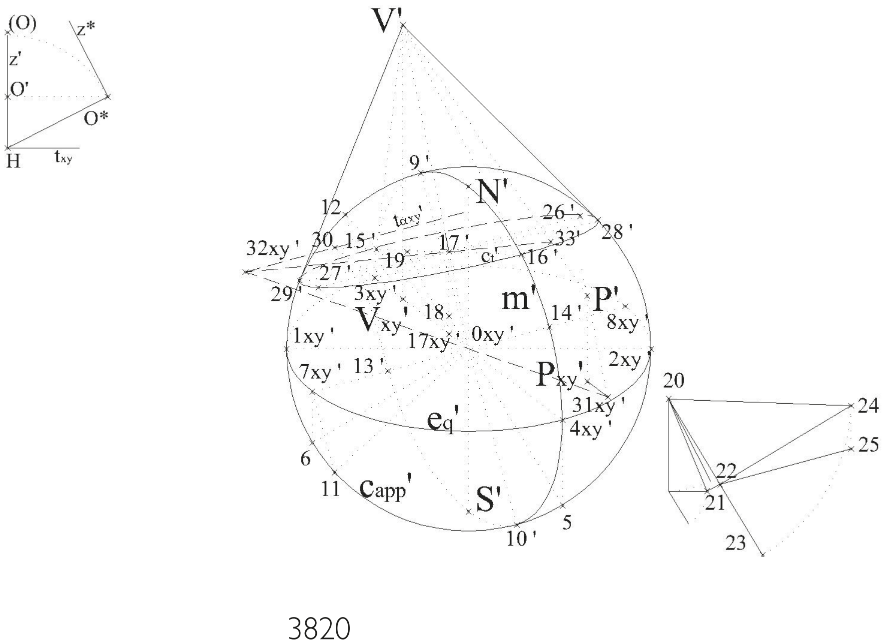


Fig. 4. Cone tangent to the sphere from an external point (image by the author).

drawn in the figure, we can easily derive the intersections on *xy* of the generators *V′-16′-16xy′* and *V′-17′-17xy′*, where *16xy′* and *17xy′* are endpoints of a diameter, as well as those of *V′-27′-27xy′* and *V′-26′-26xy′*, where *27xy′* and *26xy′* are endpoints of a chord parallel to *7xy′-0xy′-8xy′* and conjugate to the previous diameter; having a diameter and a conjugate chord, we know how to construct the ellipse *cxy′*. Given a point *P(P′, Pxy′)*, I draw through *P′* the line parallel to *V′-17′* and through *Pxy′* the line parallel to *0xy′-17xy′*; I obtain lines *31xy′-0xy′-32xy′* and *32xy′-17′-33′*; point *33′* is the intersection of line segment *P′-V′* with plane *α*, falling inside the *ct′*; I conclude that point *V* does not see point *P*.
I still consider the previous sphere *σ* (fig. 5); I mark an internal point *P1′* on the apparent contour, which has its projection on the equatorial plane in *3xy′*; I check if *P1′* is on the surface of *σ*; I rotate *0xy′-3xy′-4* into *0xy′-5-2xy′*; for point *5*, I draw the vertical line *5-6*, extend it to *5-7* parallel to *z**, and shorten it to *5-8*; according to *4-2xy′*, from point *8* I derive *P′*; on the vertical line for *P1′*, *P′* is the position of the point that belongs to *σ*; it must have a greater elevation; *P1′* is inside *σ*; *P2′* appears inside the cap but is outside *σ*. For point *P2′*, I want to find the minimum distance from the spherical surface; I can follow different paths; I find the homologous of *0xy′-3xy′*, draw a line parallel to *0xy′-3xy′* through *O′* until *txy*, connect with *O*, and draw a parallel through *0xy′*, obtaining the dimension of *0xy′-3xy′*; I measure along *z* the apparent elevation from *3xy′ to P2′*; I subtract the radius from the hypotenuse constructed with the two measurements and obtain the required distance. I can also follow the path of rotation transfer; from *P2′*, I consider *P2′-9* parallel to *4-2xy′*, then follows *9-10* and arc *10-P*2*; connecting with *0xy′*, at *P2-Q2** I obtain the sought distance.
The vertical line *r′* intersects the plane *σ* at *rxy′*; from *rxy′,* we follow *5-6-7-8-R1′*; the symmetric point *R2′* of *R1′* with respect to *rxy′* is the second intersection; if the line intersecting *σ* is *s(s′, sxy′)* with trace *Tsxy′*, the resolution path can be as follows: I consider *0xy′-11-Tsxy′,* then *11-1xy′*, and *Tsxy′-12* parallel to *11-1xy′*; *sxy′* intersects *txy* at point *13*, which is the projection of point *14* of *s′*, followed by *13-15* parallel to *z*, then *14-15*, and the arc centered at *13* with radius *13-15* gives us point *16*; line segment *12-16* is s, which intersects the circle at points *17* and *18*; through line segment *18-19* and the line parallel to *11-1xy′* for point *19*, we find *S1xy′* on *sxy′*; on *s′*, we find *S1′*, the first intersection point of line *r* with the sphere; similarly, we find *S2′* on *s′* from *S2xy′* on *sxy′*. If the line is such that *Tsxy′* is out of range, and point *13* common to *sxy′-txy* proves inconvenient, I can always resolve the intersection by considering plane *r′-rxy′*, which intersects σ according to a coplanar ellipse and intersects the line at the two required points; the line may turn out to be non-intersecting.

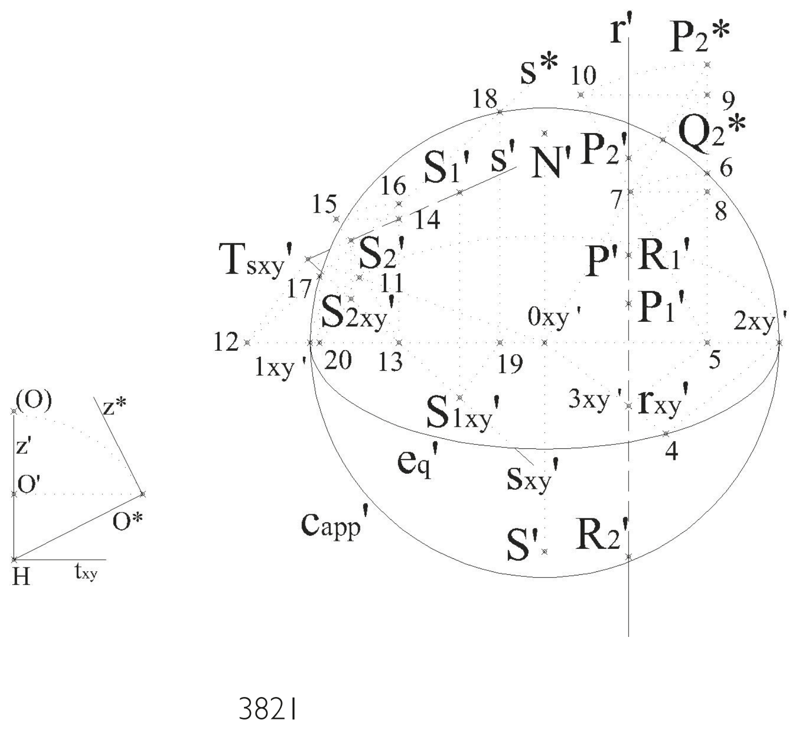


Fig. 5. Distance Point to sphere (image by the author).

Conclusions

The geometric mastery of construction methodologies ensures an adequate proficiency in the use of cutting-edge software. In turn, the use of exemplary yet rigorous methodologies guarantees a deeper understanding of the science of representation.
In this paper, a constructive methodology in orthogonal axonometry has been experimented with, which, starting from homologous shortening, allows for the rapid derivation of more complex geometric elements on the surface of the sphere. Starting from the representation of the sphere, parallels and meridians have been derived to move on to more elaborate constructions, such as circles parallel to a given meridian, the cone tangent to the sphere from a point outside the sphere, and the distance of a point from the sphere.
All constructive methods are influenced by simplifying adjustments that allow for immediate and effective processing, which becomes a key approach in the use of state-of-the-art software. Finally, several alternative solutions are proposed that may be suitable depending on the specific cases.

Reference List

Hirsch, A. (2002). Extension of the "Villarceau-section" to surfaces of revolution with a generating conic. In *Journal for Geometry and Graphics*, 6(2), pp. 121-132.

Inzerillo, L. (2012). *Assonometria e futuro.* Roma: Aracne Editrice.

Brezov, D., Kancheva, Y., Nikova, T. (2013). Non-Euclidean Descriptive Geometry in Engineering and Applied Visual Arts. In *Journal for Geometry and Graphics*, 2(2), pp. 227-236.

Koch, R., Engelhart, C. (1998). Closed space curves of constant curvature consisting of arcs of circular helices. In *Journal for Geometry and Graphics*, 2(1), pp. 17-31.

Mileikovskyi, V. (2019). A task about spheres and cones, applicable in jet theory. In *Journal for Geometry and Graphics*, 23(1), pp. 99-114.

Ratcliffe, S. (2018). *Oxford Essential Quotations*. Oxford: Oxford University Press.

Schreiber, P. (2002). Generalized descriptive geometry. In *Journal for Geometry and Graphics,* 6(1), pp. 37-59.

Sugihara, K. (2002). Laguerre Voronoi diagram on the sphere. In *Journal for Geometry and Graphics*, 6(1), pp. 69-81.

Author
Laura Inzerillo, Università degli studi di Palermo, laura.inzerillo@unipa.it

To cite this chapter: Laura Inzerillo (2025). The Study of the Past for the Overcoming of the Future. The Study of the Sphere in the Science of Representation. In L. Carlevaris *et al.* (Eds.). *èkphrasis. Descrizioni nello spazio della rappresentazione/èkphrasis. Descriptions in the space of representation.* Proceedings of the 46th International Conference of Representation Disciplines Teachers. Milano: FrancoAngeli, pp. 3815-3822. DOI: 10.3280/oa-1430-c953.

ISBN 9788835182412