MATHEMATICAL EXPERIENCES TO CONTRAST LEARNING LOSS AND SCHOOL DROP-OUT

IX Seminar "Data from and for educational system: tools for research and teaching"

edited by
Patrizia Falzetti

FrancoAngeli

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Introduction

by Patrizia Falzetti

The seminar dedicated to INVALSI data is a valuable opportunity to explore crucial issues in education. For example, over the course of its various editions, the Mathematics sessions have provided a valuable opportunity to analyse results, recurring errors, and the cognitive processes involved in solving test items in detail, thereby fostering targeted didactic reflection.

This volume collects six contributions presented at the ninth edition of the Seminar "Data from and for the educational system: tools for research and teaching" (Rome, 17-19 October 2024). Each contribution addresses some of the main challenges in contemporary Mathematics education from different yet complementary perspectives: linguistic complexity, socio-economic inequalities, the impact of the pandemic, teacher education and the risk of school dropout.

The first chapter explores how students' linguistic repertoires can support Mathematics learning in multilingual contexts. The second chapter analyses the evolution of learning loss in Mathematics between 2020 and 2024, with particular attention to disadvantaged students. The third chapter presents the results of an online survey investigating Mathematics teachers' knowledge of the national standardised INVALSI tests. The fourth chapter examines early school leaving in Italy and Europe, focusing on lower and upper secondary education. The fifth chapter describes and discusses teaching experiences aimed at fostering problem-solving processes in disadvantaged contexts. Finally, the sixth chapter analyses school dropout and the role of inclusive education in its prevention.

Overall, these contributions provide a comprehensive overview of the factors that influence the learning of Mathematics, including language, context, motivation, teaching quality, and inclusion. The volume thus provides a useful framework for reconsidering Mathematics as a formative experience that can foster participation, equity and meaning.

1. From classroom teaching practice to significant linguistic experiences: how to support Mathematics learning in heterogeneous primary school class

by Emanuela Atz, Pier Luigi Ferrari, Giovanna Mora

The development of paths with objectives common to multiple disciplinary fields (GISCEL, 1975) is the focus of the activities aimed at supporting learning in Mathematics in the primary school of a Comprehensive School in the province of Bolzano. Starting from the students' linguistic repertoires, the longitudinal path of knowledge organization is developed through both the argumentation in Mathematics problems and the exploration of the language system and the related disciplinary languages (Ferrari, 2021) in a context of endogenous multilingualism – teaching in Italian and German – and exogenous – languages of the students (Iannaccaro, 2019). Argumentation is an end and a means for the development of linguistic and mathematical skills, leveraging the different linguistic repertoires present in the classroom and proposing texts in the languages of instruction and in some of the students' L1s. The strategies used (e.g. translation, peer tutoring, in-depth study in L1) for the comprehension and production of texts can support learning with significant linguistic experiences. The latter strengthens self-esteem and awareness of one's linguistic and mathematical competence.

Lo sviluppo di percorsi con obiettivi comuni a più ambiti disciplinari (GISCEL, 1975) è al centro delle attività volte a sostenere l'apprendimento della Matematica nella scuola primaria in un istituto comprensivo della provincia di Bolzano. A partire dai repertori linguistici degli alunni il percorso longitudinale di organizzazione delle conoscenze si sviluppa sia attraverso l'argomentazione nei problemi matematici sia mediante l'esplorazione del sistema linguistico e dei linguaggi disciplinari correlati (Ferrari, 2021) in un contesto di plurilinguismo endogeno – insegnamento in italiano e tedesco – ed esogeno – le lingue degli alunni (Iannaccaro, 2019). L'argomentazione rappresenta sia un fine sia un mezzo per lo sviluppo delle competenze lin-

guistiche e matematiche, valorizzando i diversi repertori linguistici presenti in classe e proponendo testi nelle lingue di insegnamento e in alcune delle lingue ereditarie degli alunni. Le strategie adottate (e.g. traduzione, tutoraggio tra pari, approfondimento in L1) per la comprensione e la produzione dei testi possono sostenere l'apprendimento attraverso esperienze linguistiche significative. Queste ultime rafforzano l'autostima e la consapevolezza delle proprie competenze linguistiche e matematiche.

1. Introduction

The context of South Tyrol in Italy, due to its geographical position and historical background, is a compelling example of thriving linguistic diversity within a single province, with three official languages (German, Italian, and Ladin) and various dialectal variations. In this small community (534,147 inhabitants – ASTAT, 2022) the endogenous diversity has been further enriched, over the past thirty years, by people from 144 different countries, each one of them contributing with their languages and varieties to the fabric of South Tyrolean society.

Due to the complex sociolinguistic environment, in some primary schools, teachers, and educators developed paths with objectives common to multiple disciplinary fields (GISCEL, 1975) and focused some activities to support learning in maths. A phenomenon observed in some of the heterogeneous classrooms is the "summer slide effect" during the summer break. The issue of pupils forgetting during summer breaks what they had learnt at school (Bazoli *et al.*, 2022) has been extensively researched – with an increase of publications after COVID-19 pandemic. There is no full consensus on the size of knowledge loss or which type of pupils could be more affected, but there is a general agreement about the risk for children disadvantaged by race, low income, or social status. Outside of the school environment, these pupils may be exposed to less cognitively stimulating home environments during summer and learning loss contributes to the widening educational inequalities.

Therefore, an intervention in the summer holiday could be useful to reduce learning loss. The paper will address the following questions:

- How can the exploration of the language repertoires help teachers in organising disciplinary content (for instance in Mathematics)?
- How effective is argumentation in problem solving for primary school pupils to overcome the typical difficulties of a multilingual environment?
- Which (teaching) actions can be planned during the summer break to mitigate summer knowledge loss?

We provide a brief overview of the context in which the intervention was implemented with a focus on two groups during the 2023/2024 school year, comprising pupils aged 8 to 11. Next, we examine the relationship between language and Mathematics, highlighting key elements. Following this, we explore the role of argumentation in problem-solving and analyze the collected data. In the discussion, we emphasize the pedagogy of Mathematics, presenting evidence for potential interventions in multilingual classrooms.

2. The context: the schools and the language centres

2.1. The schools

The context is the primary school in a district of the Province of Bolzano, situated in a bilingual sociolinguistic context: German and Italian with their dialectal varieties are spoken by the autochthonous population in different percentages in the villages of this area. The pupils are enrolled in the schools with three languages of instruction: Italian, German and English. In Fig. 1 the example of the lesson distribution for grade 1.

<table>
<tr><th></th><th>Lessons (weekly)</th><th>Lessons
(morning: 60/50 min; afternoon: 45 min)</th></tr>
<tr><td>Italian</td><td>7</td><td rowspan="7">In Italian (language)</td></tr>
<tr><td>Mathematics</td><td>6</td></tr>
<tr><td>History</td><td>2</td></tr>
<tr><td>Science</td><td>2</td></tr>
<tr><td>Religion</td><td>2</td></tr>
<tr><td>Sport</td><td>2</td></tr>
<tr><td>Art/Music</td><td>1</td></tr>
<tr><td>English</td><td>1</td><td>In English (language)</td></tr>
<tr><td>Music/art in L2</td><td>1</td><td rowspan="3">In German (language)</td></tr>
<tr><td>Geography in L2</td><td>1</td></tr>
<tr><td>German and Maths in L2</td><td>7</td></tr>
<tr><td>Total</td><td>31</td><td></td></tr>
</table>

Fig. 1 – Lesson distribution in the three school languages for grade 1

At school pupils listen, speak, interact, and learn to read and write in three different languages. Most of them, nearly 50%, have other family languages at home too, so that we can speak of «heritage language speakers with unique profiles neither for L1 nor for L2» (Yip, 2023, AILA Conference). In this complex sociolinguistic context, teachers can work with their pupils as researchers (Loiero and Lugarini, 2019) and pupils as «young researchers» (2019, p. 132).

2.2. The language centers

The language centers are a common project between linguistic groups in the Province of Bolzano for the creation of centres for promoting the integration of pupils with migrant backgrounds. They support pupils and families with different measures and projects from their arrival through their integration into the educational settings. From 2008 to 2020 the core business of the language centers was coordinating language support teachers (for German and Italian as second language – denomination: A023-bis/ter), organising language courses, counselling about integration projects in the school and the management of the intercultural mediation service.

From 2020 onwards the role has changed from coordination to targeted consultancy, with the promotion of plurilingualism at kindergarten and school, supporting more families' involvement in the educational system. Language courses (known as L2 courses, either Italian or German depending on the educational department) are offered for all pupils, especially in the summertime before the beginning of a new school year: from 2008 onwards every summer in the last two weeks of August modules of 20 hours (2 lessons a day) per group are organised with qualified teachers. Schools have been cooperating with the language centers in several projects, activating measures to address the complex sociolinguistic fabric of its classrooms and the different groups. Some results of an activity undertaken in the experimentations during summer break are the focus of this contribution.

3. Interpretation and guidelines

In multilingual classrooms, maths and science teachers navigate three interconnected domains: Mathematics, language, and pedagogy. They have following characteristics:

- *Mathematics*: learning Mathematics shares similarities with learning a language, with Mathematics functioning as both a medium and a message (Pimm, 1987).
- *Language*: the process of learning Mathematics raises significant questions about language; however, addressing language fluency alone does not resolve the underlying challenges.

– *Pedagogy*: the task of pedagogy is defining the qualities of effective Mathematics teaching and learning within multilingual contexts.

These domains – Mathematics, Language, and Pedagogy – intersect and must be considered collectively. Research on Mathematics education and language diversity has been widely explored across various domains. This body of work encompasses a range of studies relevant to language and meaning-making (Khisty, 1995), mathematical text problems (Barwell, 2009), Mathematics teaching and culture (Civil, 2002), conversation support (Moschkovich, 2008), language policy (Setati, 2008), and bilingual teaching (Martin-Jones, 2000), among others. The role of language negotiations in multilingual Mathematics classrooms has been particularly examined in multicultural contexts, such as South Africa, where studies by Setati, Nkambule, and Goosen (Goosen *et al.*, 2011) provide evidence supporting Cummins' (1996) assertion that «students' proficiency in their first language is crucial for second language learners». These findings suggest that second-language learners should be encouraged to actively use both languages and/or their broader linguistic repertoires in all aspects of their learning process. Furthermore, classroom research investigating multilingualism and the teaching and learning of Mathematics identified tensions that can arise in multilingual classrooms (Adler, 2001). Three interrelated dilemmas were pinpointed:

– *Code switching*: whether and when to switch languages. Bose and Choudhury (2010) supported the use of both code switching and code mixing as important skills in breaking down the language barriers: code switching relates to switching between representations, and code mixing relates to changing the oral language used to assist in making connections between the representations.
– *Mediation*: the shift towards learner-centred practices and context neutrality, relying on pupils' communicative skills. Clarkson pointed out that «the competencies that the students have with both their languages are important in how well they perform mathematical tasks» (2007, p. 212).
– *Transparency*: language should be used as means of communication, clarity and access to mathematical discourse, explicit mathematical language.

Finally, language should be seen as a resource in every subject: the language of instruction and the heritage language(s) are fundamental to student's learning (Matthews and López, 2019), even in Mathematics where the connections between bilingualism and learning may be less clear.

4. Key elements and experimentation

4.1. What kind of problems?

According to the Recommendation of the Council of the European Union adopted on 22 May 2018 (p. 8) mathematical competence is defined as «the ability to develop and apply mathematical thinking and understanding to solve a range of problems in everyday situations». Therefore, contrary to scholastic and editorial tradition, problem solving is no longer aimed at applying knowledge and skills, but the latter become tools at the service of the former. According to the National Guidelines of 2012 problems «must be understood as authentic and significant questions, linked to everyday life, and not just repetitive exercises or questions to which the answer is simply remembered by remembering a definition or a rule». So, it is necessary to overcome the idea of a problem as a container of operations, in which the situation described matters little. This radically changes the characteristics that problems should have to fulfil the new requirements. First, they (and the contexts involved) should be suitable to interest and involve pupils. This means also that the question should be naturally linked to the narrative context of the problem. In other words, the question should be interesting from the viewpoint of the story, not just of Mathematics.

Second, problems should be related to pupils' experience in order not only to involve them, but also to allow them to reconstruct and represent the problem situation with a sufficient degree of awareness. This means to induce pupils to put their knowledge of facts and procedures typical of the situation into play.

Third, problems should require some modeling rather than the pure application of algorithms. This means that solution methods should not consist only in the application of some algorithm (i.e., arithmetic operations) to a set of data, but should include some other strategies, such as trial and error methods and so on.

Fourth, understanding and representing the problem situation are fundamental. The language used in the statement of the problem must also avoid unnecessary complications and all implicits that can generate or consolidate stereotypes.

4.2. Basic guidelines

Research on problem solving in Mathematics has long shown that the representation of the problem situation is a crucial step in the process of problem solving (see for example Borasi, 1984, 1986). Representing the problem situation is very different from the practices based on the extraction of numerical data from the text of the problem before any attempt at interpretation. The importance of multiple representations and of their coordination has been stressed by many researchers (such as Duval, 1995). The use and coordination of different semiotic systems has beneficial consequences at both the cognitive and the operational level. From the cognitive viewpoint it allows learners to separate a mathematical idea from a specific representation, whereas from the operational one it makes available the algorithms associated with each of the semiotic systems involved (e.g., numerical computations or graphical methods), allowing the learner to choose the most suitable ones.

So, it is of fundamental importance to encourage students to represent the problem situation in different ways. From this perspective, critical interpretation of the text becomes an integral part of problem solving. This means that reflections and discussions on the text of problems should be supported, as they not only help pupils find appropriate solution strategies but also have beneficial consequences on their linguistic competence. Moreover, they make the whole process more inclusive. The next step is to build a resolution strategy. The two phases are not clearly separated: it often happens that some parts of the text that were initially neglected are reconsidered after the first attempts to develop and compare strategies. In other words, the representation of the problem situation, contrary to what is prescribed or predicted by some models of argument, not always ends before the beginning of the resolution process, but, rathers, it is a component of it.

5. Argumentation

The argumentation practices that we encourage in our classrooms are closely related to the problem-solving setting. They are aimed not just at motivating the solutions achieved but also as a help to find them. Contrary to what is proposed by some current theories of argumentation, which take care of the product rather than of the process, we are mainly interested in the latter, and in the interactions that take place among pupils, which are much more relevant from the viewpoint of teaching and learning. Relying on static models only is useless, as an argument (outside of formal systems) is never totally explicit, since it is not feasible to define all the words employed. So, the degree of explicitation to be asked must be related to the teaching goals. No argument is ever complete, but it can always be expanded or detailed.

From the perspective of learning, the process of gradually improving an argument when some critical issues are raised is much more relevant than producing arguments that match a pre-arranged pattern. So, pupils should be guided to gradually improve their arguments, as incomplete or inaccurate arguments are better than no argument, and correctness is not a prerequisite for problem solving. The work on the text of the problems is an essential part of problem solving, as well as the selection of the relevant data, which does not necessarily precede the solution process but may be completed after some argumentation and discussion.

6. The data and the activity

6.1. The sample

Data were collected over two lessons at the end of the activities by the class teacher and a researcher, from two groups of pupils participating in the summer courses. Each group attended two daily lessons, totaling 20 hours.

- *Group 1*: 8 pupils, including 1 with Special Educational Needs (SEN). The group is a mixed ability group with pupils from different classes: in their school timetable they learn subjects in Italian, German (Geography) and English (Art or Music).
- *Group 2*: 10 pupils, including 1 with SEN. This group of mixed ability pupils had previously integrated the use of oral heritage languages and representations into their Mathematics lessons during the school year.
- *Age range*: 8 to 11 years (grades 3 to 5).
- *Language proficiency in Italian*: from A1 to A2 (CEFR).
- *Linguistic diversity*: pupils collectively spoke 9 heritage languages and each pupil reported using between 4 and 6 languages, combining both heritage and school languages.

P	L1	L2	L3	L4	L5	L6
1	Moroccan	Arabic	French	Italian	English	
2	Albanian	Italian	German	English		
3	Macedonian	Albanian	Turkish	Italian	German	English
4	(S) Spanish	Italian	German	English		
5	(S) Spanish	Italian	German	English		
6	Peruvian	Spanish	Italian	German	English	
7	Chinese	Italian	German	English		
8	Chinese	Italian	German	English		
9	Tunisian	French	Italian	English	German	
10	Moroccan	Italian	German	English		
11	Urdu	English	Italian	German		
12	Bengali	English	Tagalog	Italian	German	
13	Bengali	English	Italian	German		
14	Urdu	English	Italian	German		
15	Urdu	English	Italian	German		
16	Bengali	English	Arabic	German	Italian	
17	Albanian	English	French	Italian		
18	Albanian	English	French	Italian		

Fig. 2 – Summary of individual language repertoires

Both teachers of group 1 and group 2 are teachers appointed for this specific task in the summer (summer course in Italian L2 to support pupils with migrant background). The math task was collected by a third teacher during one session in the second week of the course.

6.2. The activity and the methodology

The activity focused on a problem solving task, culminating in the prompt: "Explain your reasoning/how you figured it out".

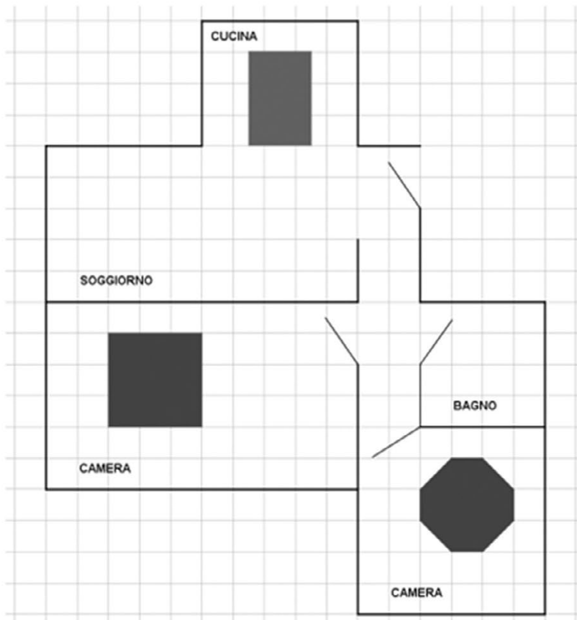
The pupils in group 1 and group 2 were exposed to the same standardized instructional procedure: they were divided into pairs or small groups and engaged in a multilingual listening comprehension activity and writing exercises. The problem text was recorded in different heritage languages and was read aloud (or listened to from a recording) in three different languages: first in one of the heritage languages of the respective two groups (Arabic, Albanian, or Spanish), followed by English, and finally in German. German and English are both languages of instruction in different subjects such as

Geography (German) or Art/Music (English). Italian and German are the languages of instruction for Maths in group 2 during the school year (6 lessons a week, 2 of which are in German).

I TAPPETI

La famiglia Rossi ha traslocato e le piastrelle del pavimento della nuova casa sono tutte di forma quadrata e della stessa grandezza.

La famiglia ha arredato la casa con nuovi mobili e con dei tappeti per renderla più bella e accogliente.



Qual è il tappeto più grande?

SPIEGA COME HAI RAGIONATO

Fig. 3 – Problem "The carpets" (Italian text)

Source: own data

After each listening of the text, the pairs/groups recorded all the words and sentences they recognized, refining and correcting their written versions after each subsequent listening.

Following each round of listening, group 1 and group 2 gathered for a plenary session to formulate hypotheses about the words that were identified and recognised as useful for comprehension. Pupils with literacy skills in their heritage language were invited to write words on the board and explain their meanings to their peers. At the conclusion of the three listening activities, the pupils received a visual representation of the problem (the map of the flat) without the Italian text. This scaffolded the information they had gathered across languages. During a final plenary discussion, the group worked together to reconstruct the problem's text in Italian, drawing on their linguistic repertoires – both the language in which they are developing fluency in (Italian L2) and their home languages.

7. Group dynamics and teacher's group management

The teacher's approach aimed to ensure that learners were simultaneously challenged mathematically and engaged linguistically. The teaching strategy was guided by two core principles:

- *Engaging with complex mathematical tasks*: by tackling interesting and challenging mathematical problems, learners developed a positive orientation towards Mathematics, enhancing their motivation.
- *Leveraging multilingual resources*: learners' home languages were deliberately and proactively incorporated into the task. The two languages of instruction (Italian and German) and the learners' home languages operated in tandem rather than in opposition. Additionally, pupils were explicitly encouraged to use any language they felt comfortable with during discussions and collaborative work.

This collaborative process fostered both linguistic exchange and collective problem-solving.

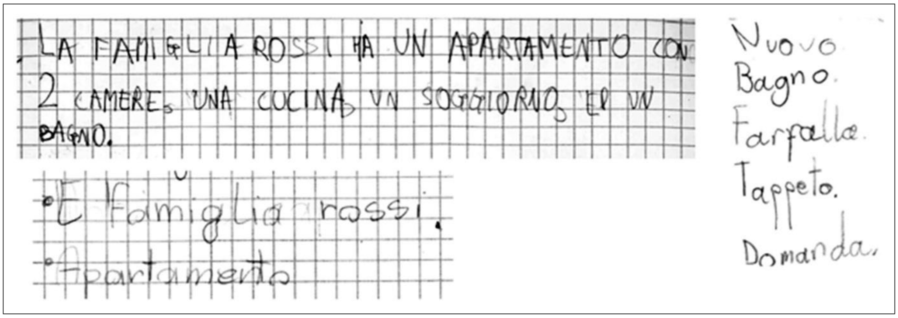


Fig. 4 – Text production after listening to the Arabic text

Source: own data

After listening to the Arabic text, for instance, the children in group 2 produced texts containing keywords such as *bathroom*, *rug*, *question*, *new*, and *butterfly* (see Fig. 4).

Some children also wrote short sentences like: “The Rossi family has a flat with two rooms: a kitchen, a living room, and a bathroom”. These written productions reveal varying levels of proficiency and lexical range in the heritage language, which appear to facilitate task comprehension. Furthermore, although the word bathroom does not occur in the original text, its inclusion by more than one group suggests that participants may have perceived a term phonetically similar to the Arabic word حمام (*ḥammām*, meaning bathroom).

During the second listening (in German), the texts became increasingly narrative, enriched with elements not present in the original text, such as “on the second floor”. This illustrates how the children’s personal knowledge base (or “personal encyclopedia”) influences their interpretation of familiar and relevant topics.

Finally, after the third listening (in English), the texts were further refined and shaped by the concluding question: “Which one is the biggest rug? Explain your reasoning”.

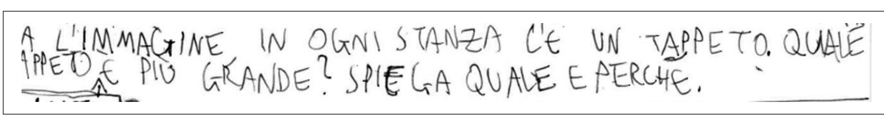


Fig. 5 – Text production after listening to the English text

Source: own data

8. Some outcomes

The pedagogical approach as outlined in paragraph 6 started from the exploration of the pupils’ language repertoires in order to organise the disciplinary content of the text. The development of activities was designed to foster simultaneous cognitive engagement in linguistic interaction with multilingual listening, multilingual written productions and mathematical reasoning. This method underscores the dual focus on developing mathematical proficiency while simultaneously supporting linguistic diversity, ensuring that students’ multilingual backgrounds are viewed as assets in the learning process.

This is evident in the following example from group 1, which had not previously engaged in any activities related to language biographies. We can notice that the short answers are given only in the target language, Italian, with no other languages supporting the argumentation, apart from the word *re(s)puesta* in Spanish. This word is given by a SEN pupil, who is looking for a suitable word in order to answer. The word *respuesta* is similar to the Italian *risposta*.

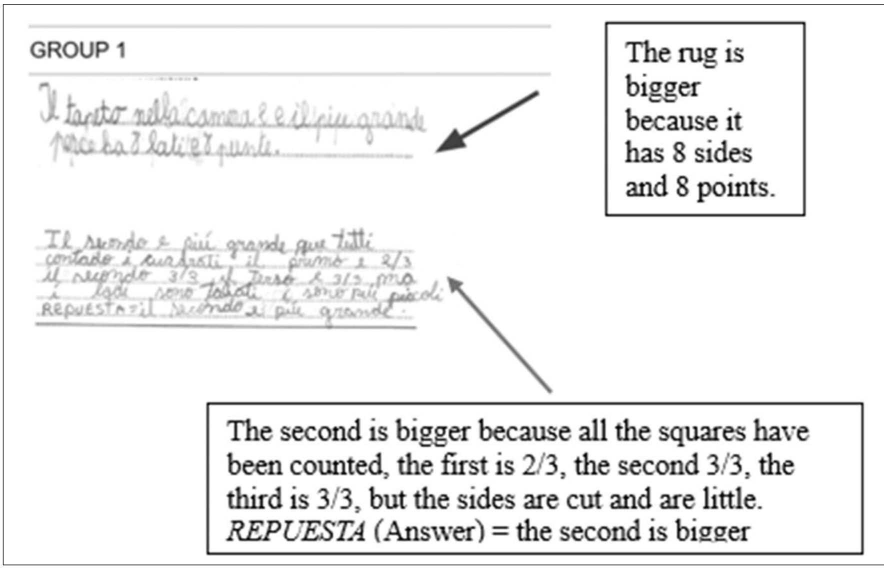


Fig. 6 – Text production class 1

Also in the sentence "il secondo è più grande *que* tutti" (the second is bigger *que – than –* the others), the SEN pupil, who is writing on behalf of the group, recalls the Spanish preposition *que* instead of *di*. These two words constitute the sole indication of a heritage language within group 1; by contrast, all other responses adhere exclusively to the target language.

The second example comes from group 2. This group had previously integrated the use of oral heritage languages and representations into their Mathematics lessons during the school year.

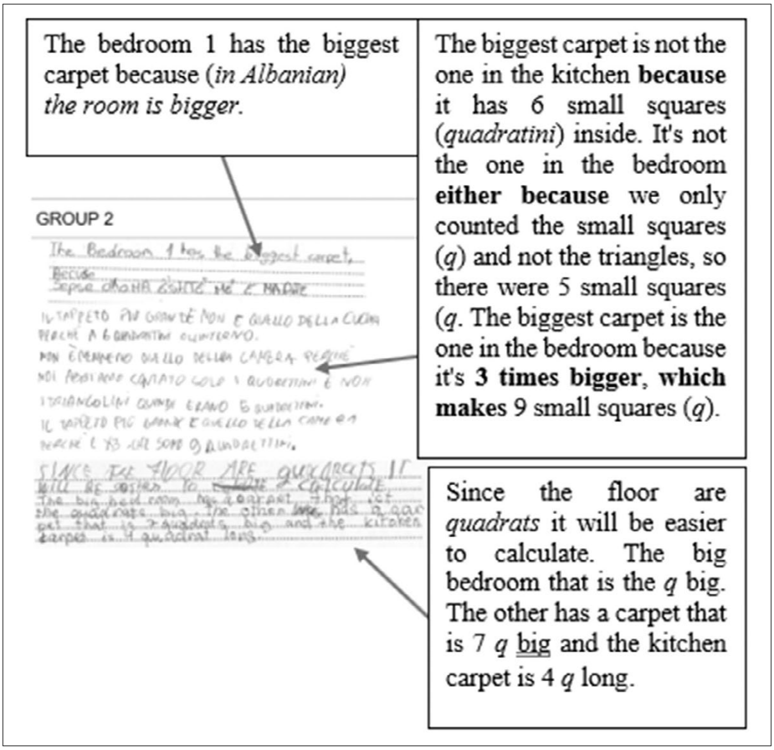


Fig. 7 – Text production group 2

The first text on the left is written by a newly arrived pupil (two weeks in Italy) with Albanian as L1 and English as second language. The pupil cannot express him/herself in the target language(s) (Italian or German), therefore it is important to support argumentation in any language he/she knows. The pupil is "languaging" in two languages (actually *translanguaging*) and this can scaffold his/her problem solving activity. The answer is probably not correct, but the effort of trying to express him/herself in L1 (Albanian) and L2 (English) to interact in the group can be seen as an important step for a newly arrived pupil. When students articulate their thoughts in their preferred languages, they engage translanguaging strategies. Research shows that these strategies promote metacognitive skills and cognitive flexibility – essential for complex reasoning and learning during other educational settings like summer courses (Pérez Fernández, 2024).

The second text is written in the classroom (target) language Italian by pupils who had been using their heritage languages and representations in their Mathematics lessons during the school year. The use of Italian can be explained by the pupils' awareness of taking part in a summer course dedicated to Italian as a second language (L2). The text reveals the use of connectors such as *because* (*perché*) and *neither* (*nemmeno*) to link sentences and substantiate the response to the prompt "explain how you reasoned" (*spiega come hai ragionato*), thereby engaging in processes of "languaging" within the target language. Using familiar language resources allows learners to articulate reasoning more clearly, supporting deeper understanding and internalization of Italian L2 structures through a scaffolded learning process.

The third text is written by two learners, with Hindi and Bengali as heritage language(s) and English as second language. The chosen language is English, relying on the fact that both were proficient in that language and could express their thoughts well enough to be understood. In the first sentence "since the floor are *quadrats* it will be easier to calculate" the pupils start their mathematical reasoning introducing it with a hypothesis as follows: *since* is the connector that indicates causality, suggesting a cause-effect relationship between the independent variable (condition) "The floor consists of squares (quadrats)" and the expected outcome or dependent variable "It will be easier to calculate". The math word learned in the classroom language – *quadrato* – is inserted as *quadrats* to complete the reasoning. Designing instructional practices that permit pupils to express themselves in the languages they find most comfortable reflects a pedagogically sound approach that can be implemented in the summer course to mitigate knowledge loss, as it leverages existing linguistic resources to scaffold learning and sustain engagement. Research on language attrition shows that lack of use leads to rapid decline in vocabulary and grammatical accuracy, especially in second languages (Jessner *et al.*, 2021).

9. Conclusion and further implications

From the classroom activity undertaken in the summer before the beginning of the new school year, we can reflect upon some results. Firstly, the exploration of the language repertoires can help teachers in knowing from which linguistic context learners come from and the collected data can support teachers in organising disciplinary content regarding linguistic accessibility of texts. Using language as a transparent resource and organising activities by engaging pupils with mathematical tasks where higher cognitive skills are required ("Tell me how you figured it out/make your reasoning explicit") can increase learners' participation and interest in Mathematics, like the first text in group 2.

Secondly, the experience described corroborates the idea that the work on the text of the problems is an essential part of problem solving. This is not related to text comprehension only but involves the grasp of the problem situation as well and helps the pupils to activate the encyclopedic knowledge they have previously developed. It also helps overcome some difficulties related to the affective dimension (e.g., pupils feeling inadequate because of their language). The arguments they produced, although sometimes incomplete, poorly expressed, and even wrong, are a fundamental starting point for further developments. Producing a wrong argument and trying to provide a justification is better than remaining silent or giving random answers.

Finally, the challenge for teachers is which actions to plan during the summer break to mitigate summer knowledge loss, especially for those pupils who may be exposed to less cognitively stimulating home environments outside of the school. We start from the assumption that there is no single, universal teaching strategy that suits all learners, situations, and contexts. Encouraging children to articulate spoken comparisons between different ideas for solving a problem not only expands their vocabulary and its practical use – by prompting them to acknowledge when others find their statements unclear – but also fosters cognitive flexibility, allowing them to reconsider how the information within a text is connected and relevant to their arguments.

Furthermore, different kinds of multilingual groups (or classrooms) may need different strategies. In class, teachers can promote pupils' everyday life experiences and their linguistic and cultural background, like getting to know the language repertoires of the class. This first step can support them in their linguistic development by focusing on the «progress in what learners already know and do» (Moschkovich, 2012, p. 91). This occurs during summer courses preceding the start of the new school year, where reflections and discussions on problems take place in a plurilingual environment. This approach not only supports pupils in identifying effective solution strategies but also enhances their linguistic skills and promotes a more inclusive learning process.

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2. Learning loss in Mathematics: the case of evolution in the period 2020-2024

by Monica Alberti, Lucia Cirina, Maria Polo

Our contribution focuses on the analysis of the phenomenon of learning loss in Mathematics in socio-economically disadvantaged situations. Specifically, we investigated the developmental trajectory of mathematical competencies in several primary school classes during the period from 2020 to 2024, also in relation to the phenomena of school dropout and disengagement. The challenging period experienced by teachers and students during the lockdown has significantly influenced school life, altering cognitive, affective-relational, and psychological aspects.

The analysis of local and national results from the INVALSI assessments of second- and fifth-grade primary school students who experienced the pandemic period provides several points for discussion regarding the contributing factors to these changes. The study shows that the developmental progression of mathematical competencies follows a different pattern for second-grade students, who experienced the lockdown in first grade, compared with fifth-grade students, who experienced the lockdown in third grade.

Il nostro contributo si focalizza sull'analisi del fenomeno del learning loss in Matematica in situazioni di svantaggio socioeconomico. In particolare, abbiamo investigato il percorso evolutivo delle competenze matematiche in alcune classi di scuola primaria nel periodo che intercorre tra il 2020 e il 2024, in relazione anche al fenomeno dell'abbandono e della dispersione scolastica. Il delicato periodo vissuto da insegnanti e alunni durante il lockdown ha influenzato la vita scolastica, modificandola negli aspetti cognitivi, affettivo-relazionali e psicologici.

L'analisi dei risultati locali e nazionali delle prove INVALSI degli alunni di seconda e di quinta primaria che hanno vissuto la fase pandemica, fornisce alcuni elementi di discussione sulle concause di tali modifiche. Lo studio mostra che il miglioramento evolutivo delle competenze matematiche ha un

andamento diverso per gli alunni di seconda, che hanno vissuto il lockdown in prima, rispetto a quelli di quinta, che hanno vissuto il lockdown in terza.

1. Introduction

The presented study focuses on the impact of lockdown on learning outcomes, analyzing the phenomenon of learning loss, which identifies the difference in knowledge, skills, or competencies measured before and after the suspension of school activities.

Research conducted on the phenomenon of learning loss highlights a correlation between learning loss and various factors: age and grade level, subject matter, family income, and socioeconomic status, as well as the learning opportunities provided by the extracurricular context. In particular, it has been shown that the effects of learning loss are more pronounced in Mathematics at higher grade levels.

A U.S. study identified the phenomenon of summer learning loss, referred to as the *tap theory* (Entwisle, Alexander and Olson, 2001). Until 2020, summer learning loss was analyzed, but since the spring of 2020, the analysis of learning loss has also extended to studying the impact of lockdown on learning outcomes.

According to this theory, during the school year, all students, regardless of their social background, can draw from the tap of resources provided by the school and make progress in their education. During the summer, the progress in learning related to the availability of opportunities offered by the school halts, negatively affecting especially students from disadvantaged backgrounds, for whom accessing books and other educational resources is more challenging.

In the current educational landscape, where inclusion and multiculturalism play a crucial role, measuring and understanding some of the contributing factors to this phenomenon is of significant importance for designing support plans aimed at combating educational poverty. Numerous studies have shown that the decrease in learning is determined by various contributing factors, not solely related to prolonged periods of school interruption; all these studies agree that these factors generally have a strong influence on motivation to learn and, consequently, on school dropout or disengagement.

The phenomenon of school dropout is the result of interactions and combinations among various factors. Three main factors have been identified: the socioeconomic and territorial context in which schools are located; the organization of the school, the student-teacher relationships, and the composition of the classroom; and the individual characteristics of students, including their predisposition and attitude toward studying and school (INVALSI, 2020).

In some research conducted through action research with teachers (Lai and Polo, 1999), the centrality attributed by teachers to relational aspects in the classroom as a strategy to address dropout was observed. Aware of the dropout issue, the teachers themselves reported that to foster self-esteem, they sought to modify their relationships with students in the classroom. Despite these changes, teachers noted that the new methodology did not improve either motivation or learning outcomes. The results remained negative, and dropout persisted. This confirmed the need to distinguish between two types of relationships that govern teacher-student communication in educational practice: the pedagogical relationship of teacher-student-knowledge, specific to the subject matter at hand, and the personal teacher-student relationship of a pedagogical nature (Polo, 2017).

Furthermore, another investigation into the factors contributing to dropout and disengagement found it important to consider that motivation to study is linked to cognitive aspects related to the pedagogical relationship of teacher-student-knowledge (Polo, Lai and Muggianu, 2021).

Other studies analyze the complex aspects of implicit and explicit school disengagement to highlight factors related to affective aspects and self-efficacy (Batini, 2023; Zan and Di Martino, 2009) that may influence the phenomenon of selective disengagement from specific subjects. We will return to the issue of implicit and explicit disengagement as a phenomenon intrinsically linked to learning loss in our conclusion.

2. Learning loss and research objectives

This study presents a longitudinal analysis of the INVALSI test results for the same students in the second and fifth grades from 2020 to 2024. The aim is to identify which areas of Mathematics have most significantly contributed to learning loss during the pandemic and post-pandemic phases. The challenges faced by teachers and students during the pandemic profoundly disrupted school life and altered cognitive, emotional, and relational dynamics within classrooms. These disruptions have potential repercussions on students' development, their negative attitudes towards Mathematics, and implicit school dropout rates.

During the pandemic and in its immediate aftermath, the equilibrium of the local educational system was severely tested, leading to a crisis in integrated education. Signs of youth distress became more pronounced, with increases in dependencies, school dropout rates, and factors contributing to social exclusion. Additionally, there was an inadequate use of digital tools, which hindered many students – particularly the most vulnerable – from fully participating in school activities and social life. For all these reasons, the model of inclusive education faced significant challenges.

The improvement initiatives implemented by the two institutions subject of the study, between the 2019/2020 and 2023/2024 academic years, such as teacher training on assessment, self-assessment, and inclusion, align with the critical themes identified in research on Mathematics education needs. An international study (Bakker, Cai and Zenger, 2021) conducted in 2020 across 44 countries revealed that the pandemic has reshaped long-term perspectives on research topics in Mathematics education. Overall, there was consensus that the pandemic amplified pre-existing issues. For instance, the shift to online learning underscored the importance of technology and prompted greater reflection on equity, as not all students had access to the necessary resources for online education. It therefore emerged, and still persists today, the need to rethink educational organization once online teaching was no longer necessary.

Consequently, the post-pandemic research challenge for teaching and learning emphasizes the need to reconsider crucial themes such as pedagogical approaches, technology, equity, attitudes, and emotional factors in Mathematics education. All educational institutions have had to grapple with these issues, seeking both short-term and long-term solutions to address the ongoing educational emergency. This research examines the experiences of two schools in Sardinia: the “Generale Luigi Mezzacapo” Comprehensive Institute in Senorbì and the “Monsignor Saba” Comprehensive Institute in Elmas, representing two distinct contexts within the region, referring to the improvement initiatives implemented by the two institutions under study, between the 2019/2020 and 2023/2024 academic years.

3. Context of the experience

This section outlines the context of the two comprehensive schools located in the province of Cagliari that are the focus of this study. The first is the Comprehensive Institute in the town of Elmas, located approximately 10 kilometers from Cagliari, which has a population of approximately 9,500 residents and comprises four educational facilities (one preschool, two primary schools, and one lower secondary school divided into 4 complexes, located within a 1 km radius) serving a total of 835 students. Originally established as a center for fishing and salt extraction, Elmas has experienced significant urban and industrial growth, marked by an increase in small and medium-sized enterprises, transportation services, and artisanal activities.

The town has undergone rapid economic, demographic, and social changes, evolving into an urban periphery with a gradual loss of its original identity. This transformation has led to various social issues, including youth distress, juvenile delinquency, homelessness, criminal activities, substance abuse, and phenomena of alienation and maladjustment.

In contrast, the State Comprehensive Institute of Senorbì is located approximately 40 kilometers from Cagliari. Unlike the Elmas Institute, it serves a broader area encompassing four municipalities (Senorbì, Barrali, San Basilio, and Suelli, divided into 12 complexes, located within a 8 km radius) with a total of twelve educational facilities – four for each level of schooling – yet with a lower student population density (740 students in a community of around 8,000 residents). The Senorbì Institute is situated in a region characterized by agricultural, pastoral, artisanal, and commercial activities. The area boasts various local infrastructures, as well as sports and cultural spaces; however, it still faces challenges such as inadequate transportation, demographic decline, and unemployment, all of which impact community life. The social fabric is diverse, with certain issues related to youth distress, which have been exacerbated by the pandemic situation. Thus, the two institutions not only exist in different geographical areas but also reflect distinct socio-economic realities and student demographics.

The two Institutes are located not only in different geographical areas but also operate within distinct socio-economic contexts and serve different student populations. Prior to 2020, both schools were equipped with technological devices in classrooms; however, these proved largely inadequate for supporting distance communication during the lockdown, with significant disparities between the two territories. During the lockdown period, following an initial phase of "instructional blackout", the Schools' PTOF (Three-Year Educational Plans) were amended to include a "Plan for Integrated Digital Teaching," in accordance with Decree-Law n. 22 of 8 April 2020, converted into Law n. 41 of 6 June 2020. Article 2, paragraph 3, of the Decree-Law establishes that «teaching staff must ensure the provision of educational activities through distance modalities, using available digital or technological tools. The decree therefore formalizes the obligation to activate distance learning through school leadership measures concerning the organization of instructional time, the allocation of technological resources, and support actions aimed at mitigating the difficulties faced by families and teachers lacking adequate connectivity». Elmas and Senorbì schools also took charge of distance learning, but they were not always successful in managing situations of family fragility: not all families had the necessary technological tools such as PCs, tablets, internet connection, cameras and, above all, the digital skills were lacking to allow the use of distance learning. In particular, at Senorbì school, especially during the initial phase, the availability of technological resources was insufficient to meet the needs of many students. This shortcoming was further exacerbated by the physical distance between school sites located in different municipalities and far from the central administration.

4. Results of second and fifth grade students in INVALSI assessments from 2019 to 2024

The period of suspension of in-person schooling lasted for three months during the 2019/20 academic year. To identify and analyze the potential effects of this interruption, which was further compounded by an additional two-month summer break, we examined the results of the INVALSI Mathematics assessments for second and fifth grade students from the 2018/2019 academic year through to the 2023/2024 assessments. The following tables indicate the year in which each grade level experienced the lockdown phase. Notably, the tables highlight the classes consisting of the same students who participated in the assessments in both second and fifth grades.

Tab. 1 – Status of second-grade classes from 2019 to 2024 in relation to the pandemic year

2018/2019	*2020/2021*	*2021/2022*	*2022/2023*	*2023/2024*
No lockdown classes	Lockdown in 1th	Lockdown in preschool	Lockdown in preschool	Lockdown in preschool

From Table 1, it is evident that the second-grade classes of the 2020/2021 academic year experienced the lockdown in their first year, while those from subsequent years had been in lockdown during one of their preschool years.

Tab. 2 – Status of fifth-year classes from 2019 to 2024 in relation to the pandemic year

2018/2019	*2020/2021*	*2021/2022*	*2022/2023*	*2023/2024*
No lockdown classes	Lockdown in 4th	Lockdown in 3th	Lockdown in 2th	Lockdown in 1th

Table 2 presents the fifth-grade classes and the year in which each experienced lockdown. Upon resuming the administration of assessments in the 2020/2021 academic year, the fifth-grade classes had undergone lockdown in the immediately preceding year, that is, in fourth grade. The fifth graders of the 2021/2022 academic year had experienced it in third grade, while those in the 2023/2024 academic year had been in lockdown in first grade.

Examining the point in their educational journey at which the fifth-grade classes had to confront a completely different mode of schooling up to that historical moment led us to reflect on the National Guidelines of 2012. This reflection aimed to consider the fundamental elements deemed essential for achieving Mathematics learning objectives. The Guidelines reference the development of the ability to connect *thinking* and *doing*, as well as the use of appropriate tools to perceive, interpret, and relate natural phenomena, concepts, and human-made artifacts, as well as everyday events. Furthermore, a fundamental element is presumed to be the laboratory environment in which students can design, experiment, explore, formulate hypotheses, discuss, and argue (*National Guidelines*, 2012, p. 49). Particularly in primary school, play and manipulation are fundamental tools for fostering communication, respect for shared rules, and the development of effective strategies in diverse contexts (*ibid.*). Laboratory-based teaching activities are an essential component of the learning process, as they foster methods such as manipulative work, cooperative learning, and other forms of interaction among peers and with the teacher, allowing students to develop cognitive, social, and metacognitive skills through discussion and active participation (ivi, p. 27). At both the Elmas and Senorbì Institutes, laboratory teaching had always been used, as a methodology advocated in the PTOF even before 2020 to pursue the educational success of each student.

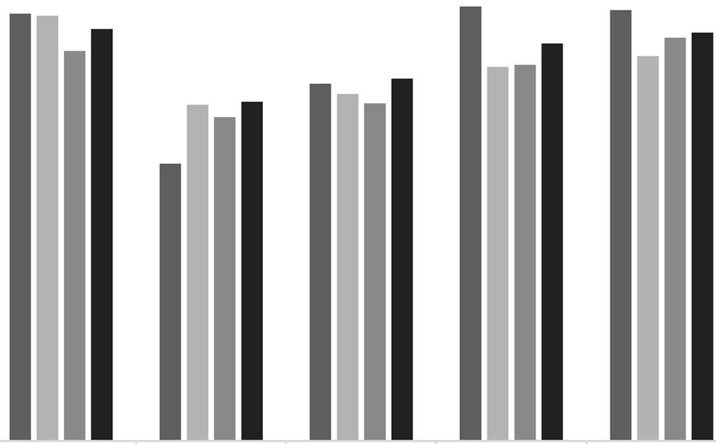
Lockdown period and the subsequent resumption of in-person school activities made it difficult, if not impossible, to conduct laboratory and play experiences, depriving students of the dynamics of discussion, cooperation, and experimentation that are the foundation of meaningful mathematical learning.

4.1. Trends of second and fifth grade classes

To identify some effects of the lockdown period on learning outcomes, we compared the responses of second and fifth-grade students in the two institutions with regional and national data. Methodologically, we conducted a longitudinal comparison of results from the pre- and post-lockdown periods to determine the trend in the percentage of correct responses across different contexts. Comparing outcomes in terms of percentages would require the specification of parameters relating to the significance of the results to examine the differences in percentages when comparing data between the two institutions and in relation to national data. In the absence of such data, from a methodological standpoint, we exclusively consider the longitudinal trend of each of the survey subjects.

Tables in Figures 1 and 2, which pertain to the response percentages of second and fifth-grade classes, present aggregated data on Mathematics test results for various academic years from the classes in Elmas, Senorbì, Sardinia, and at the national level.

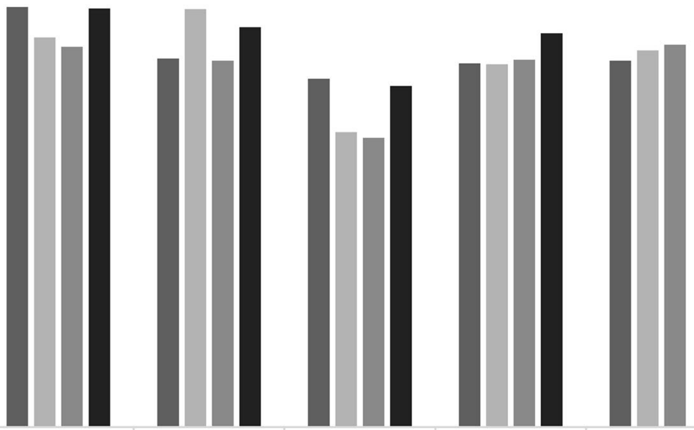
For second-grade classes, it is evident that in the 2020/2021 academic year, following the resumption of test administration after the lockdown, the percentage of correct responses experienced a significant decline compared to that of the second-grade classes in the 2018/2019 academic year. In the following years, up until the 2023/2024 school year, the trend in second-year classes is growing both at the local level of the classes of the two institutes and at the regional and national levels.



	2018/2019	2020/2021	2021/2022	2022/2023	2023/2024
Elmas	58.7	38.1	49.1	59.7	59.2
Senorbì	58.4	46.2	47.7	51.4	52.9
Sardinia	53.6	44.5	46.4	51.7	55.4
Italy	56.6	46.6	49.8	54.6	56.1

Fig. 1 – Trend of the second classes: from 2019 to 2024 – Correct answers to the INVALSI Mathematics test

If, on the other hand, we analyze the performance of the fifth-grade classes over the same time period, a different condition emerges.



	2018/2019	2020/2021	2021/2022	2022/2023	2023/2024
Elmas	58.1	51.0	48.2	50.3	50.7
Senorbì	53.9	57.8	40.8	50.2	52.1
Sardinia	52.6	50.7	40.0	50.8	52.9
Italy	57.9	55.3	47.2	54.5	

Fig. 2 – Trend of the fifth classes: from 2019 to 2024 – Correct answers to the INVALSI Mathematics test

From the graph, it is evident that the response rates of fifth-grade students experience a slight decline both at the national and local levels in the 2020/2021 academic year, followed by a more significant decrease in 2021/2022. Contrary to what one might expect, the decline among fifth graders does not occur in the year following the lockdown, as it does for second graders. In both cases considered, the percentages of correct answers in the 2023/2024 test are aligned with those of the pre-pandemic period. The trend is first decreasing and then increasing for both second- and fifth-year students, both nationally and locally.

4.2. The results of the same students in second and fifth grade

To investigate the reasons behind the identified learning loss, emerged from the comparison described above, we analyzed the performance of the same students in the INVALSI Mathematics assessments for second and fifth grades. This involves two cohorts of students who experienced the lockdown during different academic periods.

Tables 3 and 4 present the response rates recorded in the respective INVALSI assessments for second and fifth grades. National response rates have been included to facilitate a comparison with local trends.

Table 3 refers to the percentages of correct responses during the period from 2018/2019 to 2021/2022, focusing on those students who experienced the lockdown in third grade. It can be observed that the percentage of correct responses from second-grade students shows a decline compared to their performance in the fifth-grade assessment. The trend from second to fifth grade for these students, who underwent remote learning in third grade, is decreasing, both at national and local level.

Tab. 3 – Results same students in second and fiftth – Correct answers to the INVALSI Mathematics test (%)

	2018/2019 – 2nd	*2021/2022 – 5th*
Elmas	58.7	48.2
Senorbì	58.4	40.8
Italy	56.6	47.2
	Pre-lockdown	Lockdown in 3th

Table 4 presents the percentages of correct responses for the period between 2020/21 and 2023/24, which includes the classes that experienced lockdown during the first grade.

Tab. 4 – Results same students in second and fifth – Correct answers to the INVALSI Mathematics test (%)

	2020/2021 – 2nd	*2023/2024 – 5th*
Elmas	38.1	50.7
Senorbì	46.8	52.1
Italy	46.6	57.6
	Lockdown in 1th	Lockdown in 1th

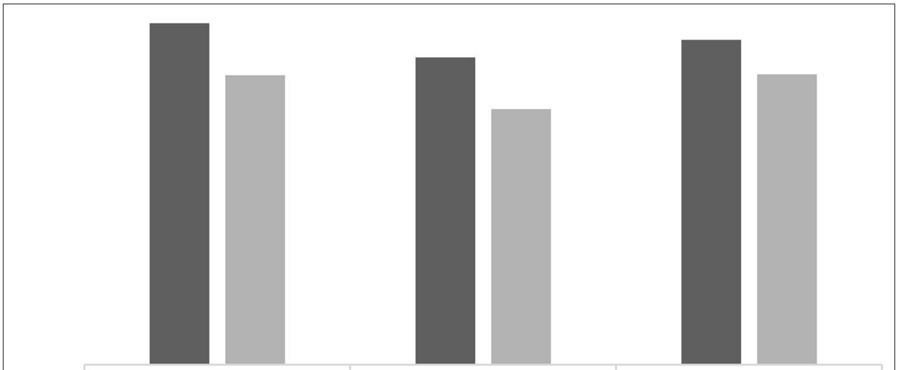
The data shows that the same students who experienced the pandemic in first grade show an increasing trend in correct answers.

We emphasize that the trend – decreasing for classes in lockdown in third grade and increasing for classes in lockdown in first grade – is similar when considering the results of the two schools compared to the national average.

This initial finding of our study indicates that the impact of the distance learning period experienced in the third grade significantly affected the learning loss resulting from the absence of regular in-person educational activities. Consequently, an epistemological issue arises regarding the mathematical knowledge at play at this educational level. In the third grade, the development of the curricular mathematical content includes both an initial deepening of the number domain and a foundational junction in the development of properties of the set of natural numbers, which leads to the expansion of numerical sets and the approach to measurement. At this level, the genesis of the pathway is established, which should lead to the objectives outlined in the national guidelines for the fifth grade.

5. The focus on domains in fifth grade assessments

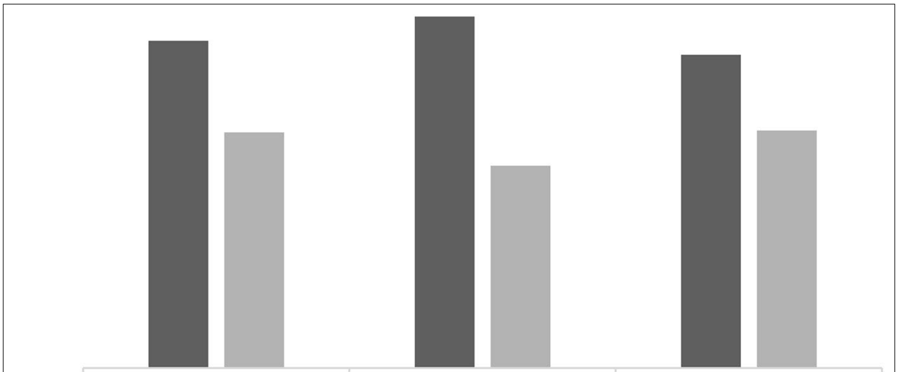
The focus we are now analyzing regarding the domains of the INVALSI assessments aims to identify which mathematical concepts have been most significantly impacted by learning loss. From a methodological standpoint, we deemed it essential to analyze the results obtained in the four domains of the INVALSI assessments for the 2021/2022 academic year in fifth-grade classes across two Sardinian institutions and in Italy as a whole.



	Elmas	Senorbì	Italy
2nd	58.9	53.0	56.0
5th	49.9	44.1	50.1

Fig. 3 – Numbers. Trend Mathematics areas second (a.y. 2018/2019) and fifth (a.y. 2021/2022) lockdown in third – Correct answers to the INVALSI Mathematics test

Prior to the lockdown, the performance of the two classes was in line with the national trend; however, for the same classes that reached the fifth grade, a slight decline can be observed compared to the national average, which similarly exhibited a negative trend. This leads us to hypothesize that local conditions, such as the presence of schools across multiple geographically distant campuses and socio-economic factors, may have a greater influence at the local level, particularly in relation to the availability and quality of online connectivity. Among the reasons for the more pronounced decline at the local level compared to the national level, we can highlight that the initial phase of the lockdown forced educational institutions into a complex and slow process of organizing remote teaching. In the two institutions considered here, substantial differences emerged based on the specific and local efficiency of online communication networks.



	Elmas	Senorbì	Italy
2nd	64.3	69.1	61.6
5th	46.3	39.8	46.7

Fig. 4 – Space and Figure. Trend Mathematics areas second (a.y. 2018/2019) and fifth (a.y. 2021/2022) lockdown in third – Correct answers to the INVALSI Mathematics test

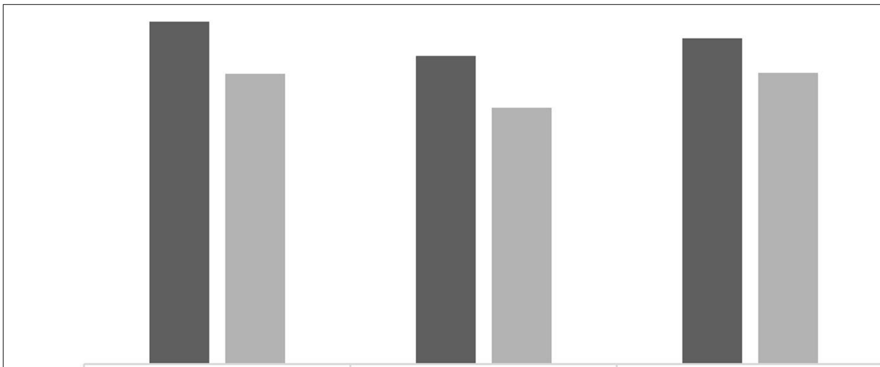
In the domain of Space and Shapes, the national data shows a significant decrease. Specifically, the percentage of correct responses drops from 61.6% in the second grade to 46.7% in the fifth grade, reflecting a decline of 24.19%, compared to a decrease of 10.71% in the Numbers domain. While one of the two institutions exhibited a more pronounced decline in accuracy, both the national baseline and the secondary institution reported consistently low performance, with correct response rates plateauing at approximately 46%.

Consequently, this study emphasizes that the downward trend in proficiency is consistent across the sampled cohorts. However, the available dataset is insufficient to provide a definitive causal explanation for the outlier observed in the first institution. We can only postulate that the discrepancy may be partially attributed to the logistical constraints and suboptimal remote communication efficacy detailed in the contextual analysis.

However, the comparison with the Numbers domain suggests that the negative trend may be attributed to the nature of the questions in the Space and Shapes domain.

This result was anticipated in relation to findings identified in the analysis of aggregated data. Such data allowed us to pinpoint the lockdown during the third grade as one of the variables that significantly contributed to learning loss. Indeed, the area of Space and Figures is the one that in third grade most requires the planning and management of in-person experiential activities. Manipulative activities, involving movement and action in physical space, as well as symbolic and graphic representation, are those that are profoundly affected by the lack of interaction with peers and teachers. Implementing these types of remote activities in the home and family context, as proposed by some teachers at both schools, can be considered a surrogate for the complexity and effectiveness of such activities conducted in the classroom using laboratory-based methodologies in which the student and social dynamics inform the teacher's decisions.

Consider the aggregate data relating to the other two areas of the INVALSI Mathematics tests: Relations and Functions and Data and Predictions.

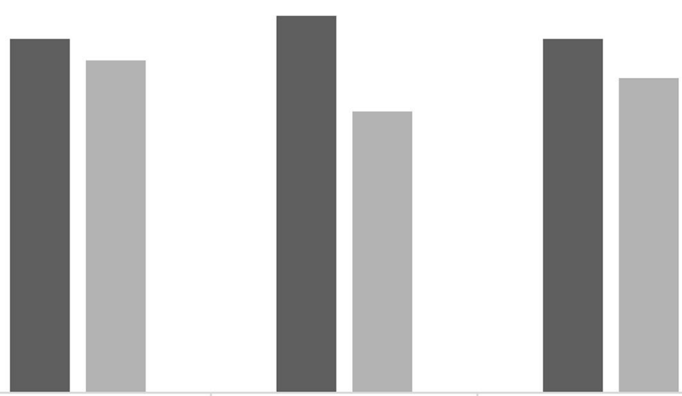


	Elmas	Senorbì	Italy
2nd	58.9	53.0	56.0
5th	49.9	44.1	50.1

Fig. 5 – Relations and Functions. Trend Mathematics areas second (a.y. 2018/2019) and fifth (a.y. 2021/2022) lockdown in third – Correct answers to the INVALSI Mathematics test

As evidenced by the data, the singular result of one of the two institutions persists, while the percentage of correct responses from the same students in the second and fifth grades remains relatively stable. The singular result observed in one of the two institutes persists; however, the percentage of correct responses among the same longitudinal cohort remains relatively stable between grade 2 and grade 5, centering around a national average of 43% for this domain.

This outcome was anticipated, as the Relations and Functions domain involves competencies rooted in mathematical knowledge that require a transition from arithmetic to algebraic thinking – a transition widely recognized as a critical juncture in the vertical development of the Mathematics curriculum. Such criticality emerges during primary education and endures throughout secondary schooling. Finally, as evidenced by the comparison with the Data and Predictions domain (see Fig. 6), a decline in correct responses persists in grade 5 compared to those provided by the same students in grade 2. The national average for this domain stands at 46%, slightly exceeding the performance observed in the Relations and Functions domain. As in the previous case, the competencies related to mathematical knowledge concerning probability and statistics are among those that pose significant didactic challenges even at subsequent educational levels.



	Elmas	Senorbì	Italy
2nd	52.1	55.5	52.1
5th	48.9	41.4	46.3

Fig. 6 – Data and estimated. Trend Mathematics areas second (a.y. 2018/2019) and fifth (a.y. 2021/2022) lockdown in third – Correct answers to the INVALSI Mathematics test

The domain-specific analysis identified grade 3 as a pivotal stage for the acquisition of competencies that are particularly critical to the vertical development of the Mathematics curriculum. To further investigate the potential contributing factors of the declining trend in correct responses, we examined specific items representative of routine classroom activities that were suspended during the forced school closures. Consequently, we conducted a detailed granular analysis, by domain, of selected items with the highest frequency of incorrect answers from the 2021/2022 National INVALSI assessments. These assessments were administered to grade 5 students who experienced the primary lockdown period during their third year of schooling. We therefore analyzed in detail, for each area, some of the items with the highest frequency of incorrect answers from the INVALSI tests taken by fifth-grade students in the 2021/2022 school year, which correspond to students who had been in lockdown in third grade.

6. Items with a high difficulty rate

We considered four questions from the Numbers area, three from the Space and Figures area, and one question from the Relations and Functions area and from the Data and Predictions area, highlighting both the results of the Mathematics tests of the fifth-year classes of the two sample institutes (Senorbì 6 classes: 76 students; Elmas 4 classes: 66 students) as well as the data at the national level.

Tab. 5 – Results of the INVALSI grade 5 sample 2022 (%)

Item D5	*Missing*	*Incorrect*	*Correct*
Italy	15.0	52.6	32.4
Elmas	4.6	62.7	32.7
Senorbì	16.3	66.5	17.2

The question had more than 50% incorrect answers and 15% missing answers at the national level and a similar trend is confirmed in the data relating to the two Institutes for which the incorrect answers exceed 60%. From a mathematical perspective, the task is not a simple addition operation; it involves converting hours to minutes or viceversa, thus representing a problem of time measurement and it is also necessary to achieve mastery of sexagesimal units of measurement.

D5. **Giovanni ha preso un treno che doveva arrivare a Roma alle ore 13:45, ma che invece è arrivato con 110 minuti di ritardo.**

A che ora è arrivato a Roma il treno preso da Giovanni?

Scrivi il procedimento che fai per trovare la risposta e poi riporta sotto il risultato.

....

....

....

Risultato: ....

Fig. 7 – Numbers, item D5 grade 5, Mathematics INVALSI test, 2022

Translation by the authors: "Giovanni took a train that was supposed to arrive in Rome at 13:45, but it arrived with a 110-minute delay. What time did Giovanni's train arrive in Rome?"

The approach to the measurement of quantities is typically developed starting from the second grade with arbitrary quantities and continues in third grade through to fifth grade with the introduction to the official system of units of measurement. Measurement activities are carried out through laboratory-based tasks, often within the context of topics related to space and shapes. Therefore, this item encompasses all the didactic issues identified earlier in the analysis of the covered areas.

Tab. 6 – Results of the INVALSI grade 5 sample 2022 (%)

Item D11	*Missing*	*A*	*B (correct)*	*C*	*D*
Italy	1.9	7.7	51.0	20.2	19.2
Elmas	1.4	9.5	46.3	28.6	14.2
Senorbì	1.3	13.1	46.1	25.0	14.5

The question D11 also presents a challenge in determining the unit of measurement. Incorrect answers C and D account for nearly 40% of the total national answer scores, and the same percentage is also maintained at around 40% at both institutes. These answers are likely determined by the common interpretation of the square as a unit representing the number line. In fact, if we consider the three squares separating the arrow from the value 900, the two results C and D are determined by considering the squares as equivalent to 10 or 1. The 1% unanswered questions and approximately 50% incorrect answers, a trend confirmed at both the national and local levels, indicate, on the one hand, a certain familiarity among fifth-grade classes with this type of question, but on the other, it demonstrates the difficulty in choosing the appropriate unit of measurement, even when measuring by counting.

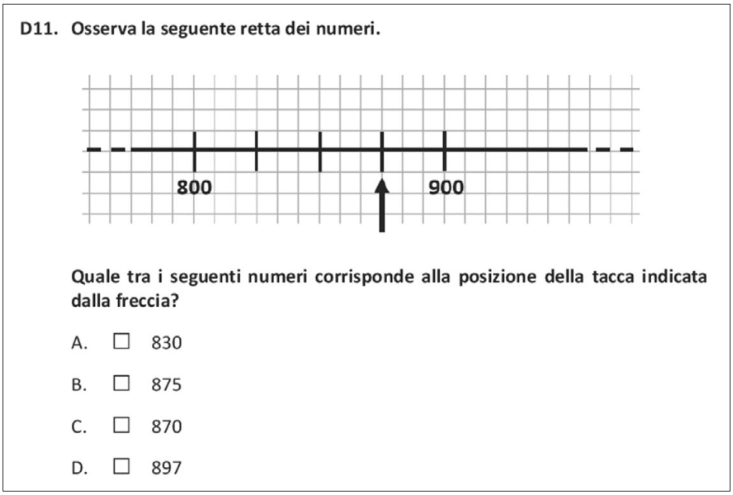


Fig. 8 – Numbers, item D11 grade 5, Mathematics INVALSI test, 2022

Translation by the authors: “Observe the following number line. Which of the following numbers corresponds to the position of the mark indicated by the arrow?”

Tab. 7 – Results of the INVALSI grade 5 sample 2022

Item D34	*Missing*	*A*	*B*	*C (correct)*	*D*
Italy	3.0	34.9	14.1	35.8	12.2
Elmas	1.4	41.0	12.2	34.8	10.6
Senorbì	7.9	39.4	14.5	30.3	7.9

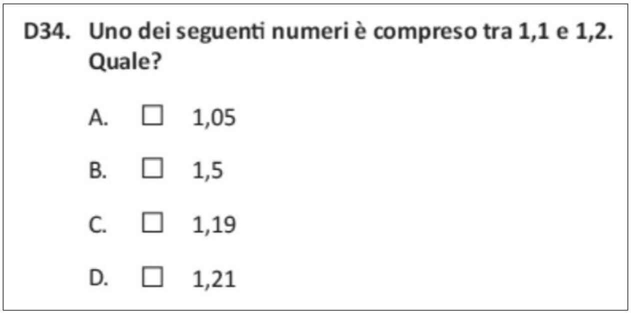


Fig. 9 – Numbers, Item D34 grade 5, Mathematics INVALSI test, 2022

Translation by the authors: “Which of the following numbers is between 1.1 and 1.2?”

Question D34 concerns the ordering of decimal numbers, known in the literature as the «epistemological obstacle» (Brousseau, 1986). More than

60% answer incorrectly, providing responses A, B, and D, while only 3% do not respond. This result confirms the intrinsic difficulty of ordering within the set of rational numbers. In particular, the incorrect answer A (1.05) has a percentage of 34.93%, which is similar to the 35% for the correct answer C (1.19), demonstrating the misunderstanding of the ordering between tenths and hundredths.

Tab. 8 – Results of the INVALSI grade 5 sample 2022 (%)

Item D30	*Missing*	*A (correct)*	*B*	*C*	*D*
Italy	3.2	42.0	13.1	5.7	36.0
Elmas	3.1	35.0	9.0	1.6	51.3
Senorbì	2.6	40.8	13.1	6.6	36.9

D30. 26 chilometri e 37 metri equivalgono a:

A. ☐ 26037 metri

B. ☐ 26370 metri

C. ☐ 2,6037 chilometri

D. ☐ 26,37 chilometri

Fig. 10 – Numbers, Item D30 grade 5, Mathematics INVALSI test, 2022

Translation by the authors: “26 kilometers and 37 meters are equivalent to: A. 26037 meters; B. 26370 meters; C. 2.6037 kilometers; 26.37 kilometers“

Finally, in the last question selected in the Numbers area, the incorrect answer D, which occurs between 36% and 51% of responses at the national and local levels, clearly illustrates the complexity of the relationship between mastery of the positional system and linguistic skills.

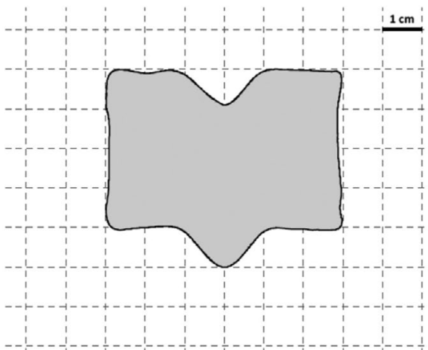
The translation from natural language of the “and” inserted in the question, understood as a comma in answer D, can be interpreted as an indicator of this difficulty. Furthermore, this question, included in the Numbers area, also contains a significant connection to the treatment of units of measurement of length and therefore to equivalences.

Furthermore, this question, included within the topic of Numbers, also shows a significant connection to the treatment of units of length and, therefore, to equivalences.

Tab. 9 – Results of the INVALSI grade 5 sample 2022 (%)

Item D6	*Missing*	*A*	*B (correct)*	*C*	*D*
Italy	2.5	31.7	37.9	13.6	14.3
Elmas	4.5	31.9	33.5	19.0	11.1
Senorbì	9.2	29.0	32.9	11.8	17.1

D6. Osserva la seguente figura.



Quale tra le seguenti misure rappresenta la stima migliore dell'area della figura?

A. ☐ 20 cm^2

B. ☐ 24 cm^2

C. ☐ 26 cm^2

D. ☐ 30 cm^2

Fig. 11 – Space and Figures, item D6, grade 5, Mathematics INVALSI test, 2022

Translation by the authors: "Observe the following figure. Which of the following measurements is the best estimate of the area of the figure?"

In this item, there is also a measurement of quantities, specifically related to area, which is addressed for the first time in the third grade of primary school. Approximately 31% of students, at both local and national levels, provided an incorrect response by selecting Option A. This error stems from calculating the perimeter by treating the grid square side as a uniform unit of measurement, even when it represents the segment approximating the square' s diagonal. For this item, correct response rates ranged from 37.9% nationally to 33% locally, further confirming the perimeter-area conflict widely documented in the literature. Moreover, the mathematical task requires "estimation", a concept of considerable complexity for this age group.

Developing a conscious understanding of estimation necessitates hands-on manipulative activities involving concrete objects, as well as managing the dialectic between different symbolic representations of rational numbers.

Tab. 10 – Results of the INVALSI grade 5 sample 2022 (%)

Item D20a	*Missing*	*A*	*B*	*C (correct)*	*D*
Italy	3.8	13.7	6.9	66.2	9.4
Elmas	1.4	18.5	1.4	68.3	10.4
Senorbì	5.3	21.0	9.2	56.6	7.9

Item D20b	*Missing*	*Incorrect*	*Correct*
Italy	14.5	51.0	34.5
Elmas	13.7	72.9	13.4
Senorbì	26.2	56.8	17.0

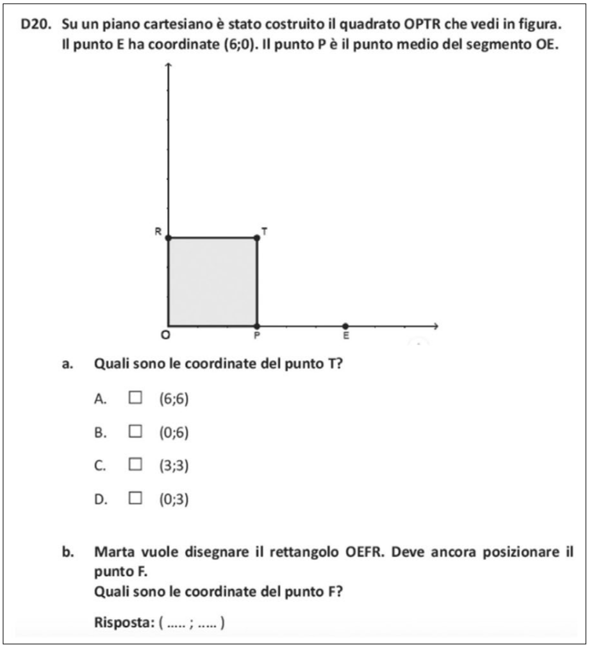


Fig. 12 – Space and Figures, items D20a-D20b grade 5, Mathematics INVALSI test, 2022

Translation by the authors: "On a Cartesian plane, the square OPTR has been constructed, as shown in the figure. The point E has coordinates (6;0). The point P is the midpoint of segment OE. What are the coordinates of point T?
Marta wants to draw the rectangle OEFR. She still needs to place point F. What are the coordinates of point F?"

The item involves the application of the Cartesian coordinate system in both a reading task (D20a) and a construction task (D20b). National correct response rates for item D20a reached 66.22%, which is comparable to the 68% and 58% observed at the local level. This mathematical task requires students to identify the coordinates of a given point by making an inference based on a figure that lacks both a grid and an explicit unit of measurement. In contrast, the correct response rates for item D20b were 43% nationally, falling significantly to between 13% and 17% at the local level. In this instance, the mathematical task requires the construction of a point not previously provided. Alongside the competency of representing a rectangle, this item necessitates the strategic selection and application of a unit of measurement.

Tab. 11 – Results of the INVALSI grade 5 sample 2022 (%)

Item D12	*Missing*	*A*	*B*	*C (correct)*	*D*
Italy	1.2	12.3	34.6	39.3	12.6
Elmas	0.0	16.2	31.0	37.5	15.3
Senorbì	5.2	15.8	38.2	30.3	10.5

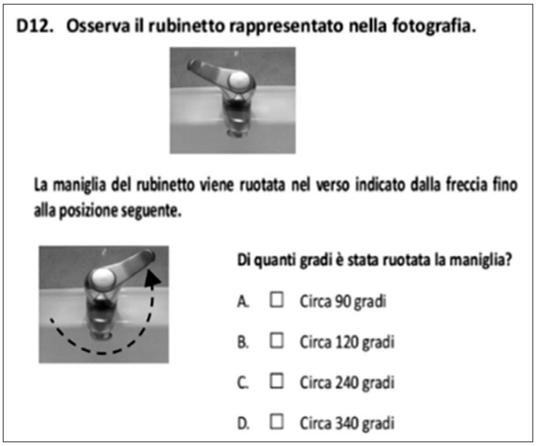


Fig. 13 – Space and Figures, item D12 grade 5, Mathematics INVALSI test, 2022

Translation by the authors: “Observe the faucet represented in the photograph. The faucet handle is rotated in the direction indicated by the arrow until it reaches the following position. By how many degrees was the handle rotated? A. About 90 degrees; B. About 120 degrees; C. About 240 degrees; D. About 340 degrees”

This item requires the application of angular measurements to solve a real-world problem, involving an implicit reference to angular units. Such units are rarely introduced in grade 3 of primary school and are typically the subject of explicit instruction at the lower secondary level. Nationally, the correct response rate was only 39.37%, with local results ranging between 30% and 37%. Notably, across the three samples considered, incorrect responses referring to a 120-degree rotation accounted for 31% to 38% of the total, confirming the challenges in achieving mastery of angular measurements at this educational stage. Even the use of a real-life context – specifically a daily experience related to rotational motion – proved insufficient for the stable acquisition of competencies in angular measurement. The lack of experimental classroom activities during the period of distance learning may be considered a contributing factor to the high frequency of incorrect responses observed in this measurement-related task.

Tab. 12 – Results of the INVALSI grade 5 sample 2022 (%)

Item D8	*Missing*	*Incorrect*	*Correct*
Italy	5.4	51.0	43.6
Elmas	3.5	74.5	22.0
Senorbì	18.4	40.8	40.8

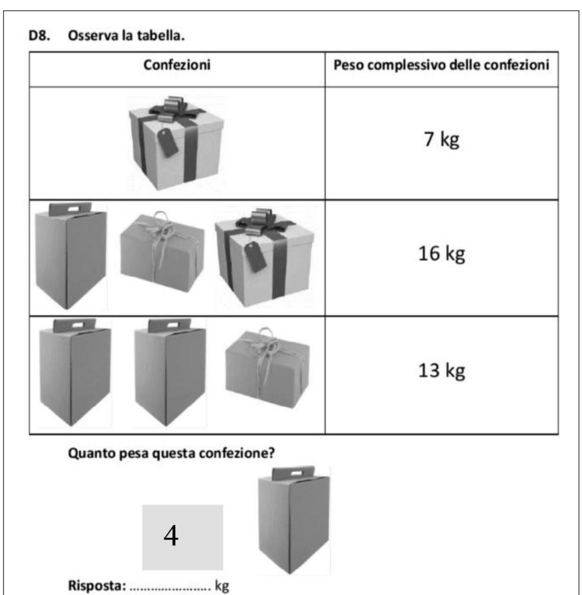


Fig. 14 – Relations and Functions, item D8 grade 5, Mathematics INVALSI test, 2022

Translation by the authors: “Observe the table. Packages/Total weight of the packages. What is the weight of this package?”

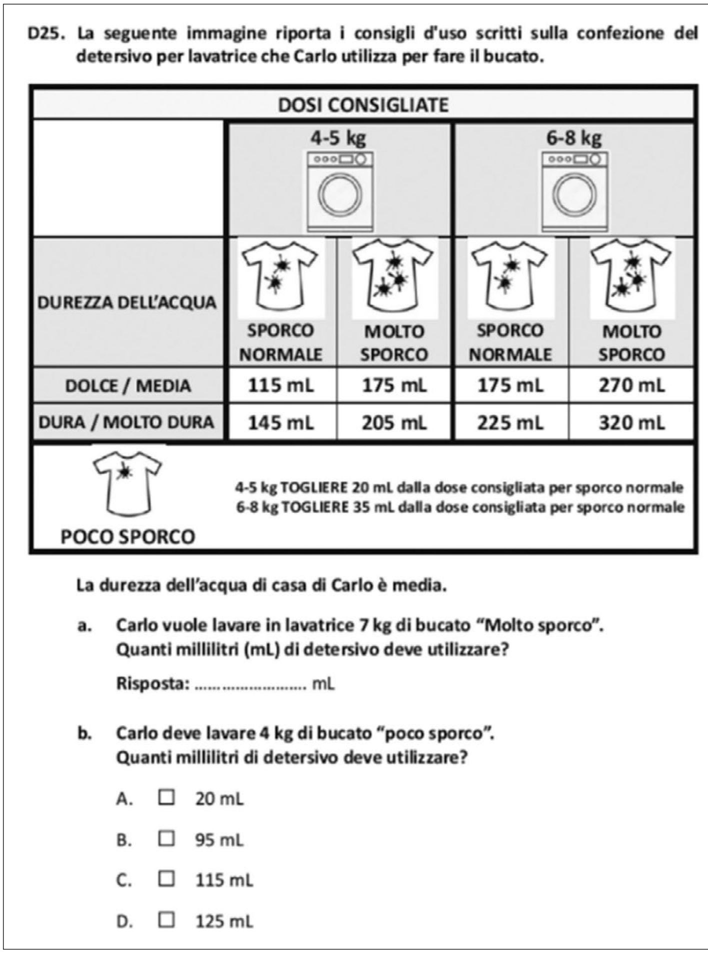
The question regarding the domain of Relations and Functions D8 represents a real-world problem that involves the measurement of quantities, in this case, kilograms, and the transition from arithmetic thinking to algebraic thinking. With 56% of incorrect or missing responses at the national level, peaking at nearly 78% locally, the data confirms the significant difficulty of this item involving algebraic reasoning. This type of problem is particularly suited for classroom activities that simulate the context of the problem statement and facilitate its resolution through a hands-on laboratory approach.

This item, within the Data and Predictions domain, investigates the ability to extract information from a complex table. The difficulty of the task is further compounded by the requirements of non-continuous text comprehension and the interpretation of data presented in tabular form. The error rate, which exceeds 50% for both items D25a and D25b at both national and local levels, is indicative of these specific challenges. Finally, we observe that the incorrect responses in the analyzed items can also be correlated with an incomplete conceptualization of mathematical principles regarding the measurement of quantities. All items considered show national correct response rates ranging from a minimum of 32% to a maximum of 66%, while local rates fluctuate between 17% and 68%. This confirms the findings, well-documented in the literature, regarding the difficulties associated with the approach to and construction of competencies in measuring quantities. The learning loss during the pandemic, exacerbated by the reliance on remote teaching, underscores the importance of an approach to measurement grounded in experimental teaching activities linked to practice and manipulation. Indeed, due to the lockdown in the third grade, such activities were compromised for these fifth-grade classes.

Tab. 13 – Results of the INVALSI grade 5 sample 2022

Item D25a	*Missing*	*Incorrect*	*Correct*
Italy	7.0	50.6	42.4
Elmas	0.0	52.8	47.2
Senorbì	6.6	40.8	52.6

Item D25b	*Missing*	*A*	*B(correct)*	*C*	*D*
Italy	3.2	13.2	43.5	29.9	10.2
Elmas	0.0	10.3	46.4	34.3	9.0
Senorbì	5.3	18.4	44.7	23.7	7.9



D25. **La seguente immagine riporta i consigli d'uso scritti sulla confezione del detersivo per lavatrice che Carlo utilizza per fare il bucato.**

DOSI CONSIGLIATE				
	4-5 kg		6-8 kg	
DUREZZA DELL'ACQUA	SPORCO NORMALE	MOLTO SPORCO	SPORCO NORMALE	MOLTO SPORCO
DOLCE / MEDIA	115 mL	175 mL	175 mL	270 mL
DURA / MOLTO DURA	145 mL	205 mL	225 mL	320 mL
POCO SPORCO	4-5 kg TOGLIERE 20 mL dalla dose consigliata per sporco normale 6-8 kg TOGLIERE 35 mL dalla dose consigliata per sporco normale			

La durezza dell'acqua di casa di Carlo è media.

a. **Carlo vuole lavare in lavatrice 7 kg di bucato "Molto sporco".**
Quanti millilitri (mL) di detersivo deve utilizzare?

Risposta: .... mL

b. **Carlo deve lavare 4 kg di bucato "poco sporco".**
Quanti millilitri di detersivo deve utilizzare?

A. ☐ 20 mL

B. ☐ 95 mL

C. ☐ 115 mL

D. ☐ 125 mL

Fig. 15 – Data and Estimated, items D25a-D25b grade 5, Mathematics INVALSI test, 2022

Translation by the authors: "The following image shows the usage instructions written on the package of laundry detergent that Carlo uses for washing clothes. Recommended doses: water hardness; normal dirty; very dirty; normal dirty; very dirty. Soft / medium; Hard/very hard. Slightly dirty:4-5 kg remove 20 ml from the recommended dose for normal dirt; 6-8 kg remove 35 ml from the recommended dose for normal dirt.
a. Carlo wants to wash 7 kg of "Very dirty" laundry in the washing machine.
How many milliliters (ml) of detergent should he use?
Answer: ____________. ml
b. Carlo needs to wash 4 kg of "Slightly dirty" laundry.
How many milliliters of detergent should he use?"

7. Discussion of results

Our study analyzed several aspects of the learning loss phenomenon in Mathematics within socio-economically disadvantaged contexts. This was achieved by comparing national INVALSI test trends with the results of two schools in Sardinia: the "Generale Luigi Mezzacapo" Comprehensive Institute in Senorbì and the "Monsignor Saba" Comprehensive Institute in Elmas. We considered two distinct realities within the Sardinian region to investigate potential contributing factors of learning loss stemming from the pandemic and distance learning, also in relation to territorial characteristics. Specifically, we investigated the evolutionary path of mathematical skills in several primary school classes between 2020 and 2024.

The study shows that the evolutionary process of mathematical competencies follows different trends for second-grade students (who experienced the lockdown in first grade) compared to fifth-grade students (who experienced the lockdown in third grade). Indeed, the same students who were in second grade in 2018/2019 and in fifth grade in 2021/2022 achieved lower percentages of correct answers compared to the tests taken three years prior. The trend from second to fifth grade for these students, who experienced distance learning during their third year, is decreasing at both national and local levels. Conversely, the students who were in second grade in 2021/2022 and in fifth grade in 2023/2024 (having experienced the lockdown in first grade) achieved higher percentages of correct answers, showing an increasing trend. Overall, at both national and local levels, we identified a similar trend in Mathematics test results for second and fifth grades during the 2019-2024 period: initially decreasing, and then increasing starting from the 2022/2023 school year.

These results confirm the impact of the lack of in-person activities and laboratory practices on the construction of meaningful and stable mathematical knowledge and skills, as evidenced by the analysis of specific content strands and individual items. The impact of the distance learning period during the third grade on the decrease in learning is particularly significant. Specifically, comparing the percentages of correct answers in the Space and Figures (Geometry) strand, the decline between second and fifth grade is more pronounced than in other areas. Considering the third-grade geometry curriculum, it is understandable that the lack of opportunities to use tools and engage in practical, manipulative, and laboratory-based activities – as well as peer discussion – hindered learning. The interruption of in-person schooling, the near-total absence of peer interaction, and the marginal role teachers were forced to play likely contributed to this learning loss. This resulted in the failure to implement the teacher's crucial role in triggering devolution of learning processes, as well as the students' lack of autonomy in the social construction of collective and individual learning paths.

On the research front, it would be interesting to continuously monitor INVALSI data from third-grade secondary school students at the end of the first cycle of education in comprehensive institutes. These classes would include students who took the fifth-grade INVALSI tests. This analysis would provide clearer insights into potential indicators of early implicit dropout. It would also be useful to analyze the influence of the composition of each class, considering native students, students with special educational needs (BES), and those with or without compensatory and/or dispensative tools, on the results achieved by the student population.

Mathematics education research has highlighted the importance of stimulating students' self-efficacy regarding Mathematics at the primary school level (Di Martino and Zan, 2011). When this result is related to the effects of implicit dropout on both total dropout and school abandonment, it becomes essential to start from primary school for a longitudinal and developmental analysis. Also, through INVALSI data, an analysis examining the learning trends of individual cohorts over time, well in advance of the end of secondary school, could be effective for developing basic mathematical competencies for all students.

8. Conclusions

In our analysis, we highlighted how the absence of hands-on teaching practices has had a significant impact on students' learning outcomes, compromising essential skills necessary for the understanding of fundamental mathematical concepts. Furthermore, by expanding our reflection on the elements contributing to school dropout, we can observe that this phenomenon has roots as early as the first cycle of education. In this regard, INVALSI employs the term *implicit dropout* to describe a phenomenon that is less evident than school abandonment and other more manifest forms of dropout, such as prolonged absences or grade retention. Implicit dropout refers to students who complete their educational journey without having acquired the fundamental competencies in the subjects assessed by INVALSI.

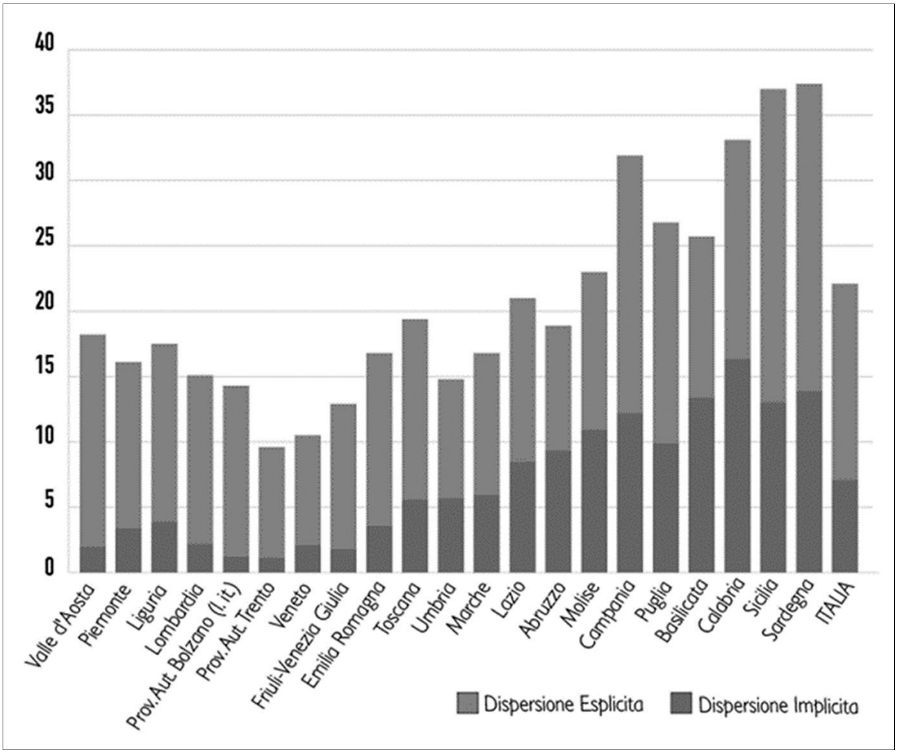


Fig. 16 – Total school dropout rates, percentage values, INVALSI data processing 2014-2019

In the graph it is possible to observe the overall effective rate of dropouts, allowing us to capture, albeit partially, an aspect of the phenomenon that has eluded official statistics. This is because it includes the segment of students who, despite obtaining a secondary school diploma, fail to acquire the fundamental competencies that such a qualification would entail (Ricci, 2019).

Monitoring the phenomenon of implicit dropout starting from primary school is essential, although it is complex to quantify at this educational level. Indeed, by retracing the phases of our study through a longitudinal analysis of the same students' results, it becomes clear why INVALSI itself advocates for the necessity of identifying standardized assessments linked to the objectives outlined in the National Guidelines, which can provide comparable data based on uniform testing for all.

Through the analysis of the lockdown experience in these two Sardinian schools, we aimed to investigate the contributing causes of learning loss related to the pandemic and distance learning, including territorial factors.

Data analysis suggests that territorial characteristics had an impact, particularly regarding deficient remote communication. However, this influence was less significant than the difficulties observed regarding mastery of the various strands of the Mathematics curriculum. Both schools sought to address post-pandemic challenges by renewing digital tools, reshaping didactic plans, and updating assessment processes. Furthermore, they addressed the psycho-emotional emergency by promoting psychological and educational support, such as counseling services and the presence of school psychologists. However, it remains essential to consider the heterogeneity within classes, which continues to pose a challenge for our inclusive school model. A commitment is needed to build an educational system capable of responding to diverse student needs, promoting a positive attitude toward Mathematics, and deploying targeted, inclusive strategies to guarantee the right to quality education for all.

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3. Results of INVALSI standardized assessment as a tool for professional development of Mathematics teachers

by Maria Chiara Cibien, Marta Saccoletto, Carlotta Soldano, Camilla Spagnolo

This study investigates Mathematics teachers' knowledge regarding the INVALSI national standardised tests. The sample consists of 131 in-service and pre-service teachers of primary, lower secondary, and high secondary school. An online questionnaire was used to collect data, and the analysis of teachers' responses is presented, particularly regarding their interpretation of student mistakes. The main theoretical framework used for the analysis is the *Mathematics Teacher Specialised Knowledge* Model (Carrillo-Yañez *et al.*, 2018). Findings highlight difficulties in identifying common student mistakes at the national level, with differences emerging based on school level and teaching experience. These insights will inform the design of an innovative teacher professional development model within the national project "Mathematics standardised assessment as a tool for teachers' professional development".

Questa ricerca indaga le conoscenze degli insegnanti di Matematica in relazione alle prove standardizzate nazionali INVALSI. Il campione di riferimento consiste di 131 insegnanti in servizio e futuri insegnanti di scuola primaria, secondaria di primo grado e di secondo grado. Per la raccolta dei dati è stato somministrato un questionario online, di cui verrà presentata l'analisi delle risposte relativamente all'interpretazione degli errori degli studenti. Il principale quadro teorico di riferimento utilizzato per l'analisi è il modello Mathematics Teacher Specialized Knowledge *(Carrillo-Yañez* et al.*, 2018). I risultati evidenziano difficoltà nell'individuazione degli errori più comuni degli studenti a livello nazionale, con differenze riscontrate in base al grado scolastico considerato e all'esperienza di insegnamento. Queste evidenze contribuiranno alla progettazione di un innovativo modello di formazione professionale per gli insegnanti, nell'ambito del progetto nazio-*

nale "Mathematics standardised assessment as a tool for teachers' professional development".

1. Introduction

The European Parliament and the Council launched a recommendation on lifelong learning in 2006 in which Member States are invited to promote key competences for all citizens in their lifelong learning strategies.

There is serious worrying data coming out from international research focusing on Mathematical competences. OECD data show that students who are fifteen years old and have reached Level 1 acquire limited mathematical knowledge on their own, which they are only able to apply in some familiar situations. Referring to INVALSI data of 2022, a similar situation occurs in Italy. Already at the end of the last century, the Curriculum and Evaluation Standards for School Mathematics (Mathematical Sciences Education Board, 1989) and the *Professional Standards for Teaching Mathematics* (NCTM, 1991) have promoted a vision for teaching and learning that has growing support from the Mathematics education community. There is increasing recognition that without carefully planned professional development programs, the chance of widespread implementation of this vision of Mathematics is small. Support is essential for practicing teachers who express an interest in teaching in a way described in the Standards, with all that such teaching implies in terms of content, pedagogy, and assessment.

The topic of teacher professional development is increasingly gaining ground in the field of educational research, also in Mathematics education.

2. The national research project

The project "Mathematics standardised assessment as a tool for teachers' professional development" (MaSt) stems from a firm and shared conviction that standardized assessments, used in a formative perspective, can become a potentially revolutionary element for the educational and didactic function of school systems. In a democratic school perspective (equity and quality together), we embraced the Bloomian belief (Bloom, 1972) that all students can learn basic skills in each topic, and we trust in the link between the development of students' skills and the teachers' professional development paths. Literature highlights that schools show an evident fragility with respect to developing an effective ability to promote good mathematical competences. It is well known that teachers have difficulties in adopting valid practices for individualized teaching techniques in order to be able to pursue quality and equity for the results of every and each one of the student population, but also in finding strategies to improve the learning for the fundamentals (Stipek *et al.*, 2001). The possibility for a teacher to use in his classroom the insights on mathematical learnings provided by the standardized tests – through their theoretical frameworks, the released items with the interpretation of macro-phenomena observed, the valuation of their results – in a formative way is an important opportunity to renew the praxis in teaching and assessment.

The use of standardized surveys to substantiate the formative assessment practices in the classroom is of high priority and of great importance. The impact of standardized assessment is traditionally a top-down impact: the results of the surveys affect public opinion; this moves the policy makers acting on the structure of the system at different levels (curriculum, teacher training, recruitment). Only in the final phase, the surveys and their results influence teachers' local action. The innovative idea of our Project is to provide instruments for a bottom-up approach in which theoretical frameworks/test/results are used immediately as a tool by teachers, to accomplish formative assessments and individualized teaching (from the perspective of quality and equity of students' achievement).

To make this approach fruitful some conditions should be fulfilled: first, there must be a link between assessments and curriculum (Mons, 2009; Meckes and Carrasco, 2010; Ferretti *et al.*, 2018). Second, this link should be perceived by the teachers, since assessment is a key component of teachers' identity (Hannula *et al.*, 2016; Ferretti *et al.*, 2021).

In Italy, the first of these conditions is met: as a matter of fact, the national system of standardized assessment (INVALSI) framework and tests are aligned with National Guidelines goals (INVALSI, 2018). There is a need for tools to develop educational activities to achieve the second condition as well. From this perspective, the main purpose of our Project is to design a model for teachers' professional development, consisting of theoretical and operational tools, starting from the resources provided by the INVALSI national assessment tests.

3. Theoretical framework

The research design framework considers three main components and results from the integration of three established theories.

We assume that the model builds on the principles of Jaworski' s *Community of inquiry* (2006), encouraging participants to critically reflect on their practices while engaging in them, question what they do as they do it, and explore new aspects of practice. That means that, in our conception, inquiry is intended as a "way of being" in which teachers develop a specific attitude that defines their way of acting, thinking, learning and teaching.

We also referred to the *Mathematics Teacher Specialized Knowledge model* (MTSK) by Carrillo-Yañez (Carrillo-Yañez *et al.*, 2018), that coordinates two broad areas of knowledge, *Mathematical Knowledge* (MK) and *Pedagogical Content Knowledge* (PCK) by placing teacher beliefs at the centre (Fig. 1). MK is the knowledge possessed by a Mathematics teacher in terms of disciplinary knowledge within an educational context and consists of three subdomains:

– *Knowledge of Topics* (KoT), that includes the knowledge of mathematical definitions, properties, representations, etc.
– *Knowledge of the Structure of Mathematics* (KSM), that involves examining elementary Mathematics from a more advanced perspective, as well as understanding how to switch and connect activities across different mathematical domain's.
– *Knowledge of Practices in Mathematics* (KPM), that regards having the ability to prove, justify, make deductions and inductions, provide examples, etc.

PCK is the knowledge related to Mathematics content in terms of teaching-learning processes and is composed of:

– *Knowledge of Mathematics Teaching* (KMT), that comprehends the knowledge of the Mathematics teaching theories, resources, technologies, strategies for introducing and representing mathematical content and concepts.
– *Knowledge of Features of Learning Mathematics* (KFLM), that comprises the knowledge about how students might act and interact with Mathematics, considering also affective aspects.
– *Knowledge of Mathematics Learning Standards* (KMLS), that consists of the knowledge of national curricula and international documents for every school level.

Beliefs about Mathematics and its learning and teaching are at the centre of the model precisely to emphasize the reciprocity between beliefs and knowledge domains. The MTSK model allowed us to frame the teachers' responses to our research questionnaire.

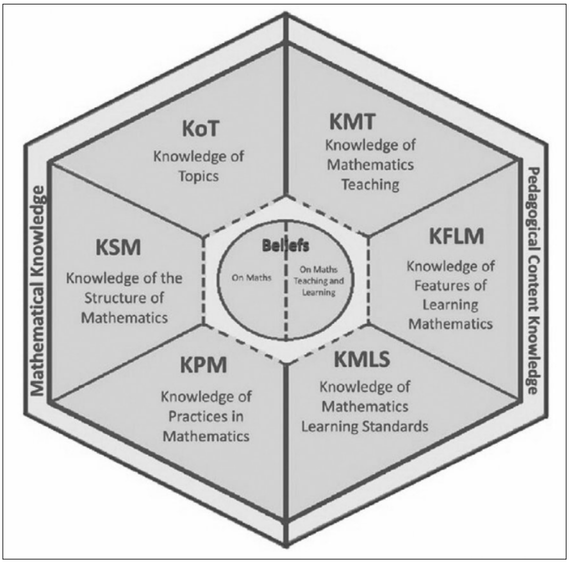


Fig. 1 – MTSK Model

Source: Carrillo-Yañez *et al.* (2018), p. 34

Finally, the third component of our theoretical framework is the *Meta-didactical Transposition Model* by Arzarello and colleagues (2014). This model focuses on the practices and dynamics that emerge throughout teacher training programs, and enables the description and interpretation of variables in teacher education processes, as well as the analysis of their mutual relations and developments. In the meta-didactic transposition process, educational practices and theoretical purposes are shared between teachers and trainers. The term "meta-didactics" emphasises that these practices involve reflections on teaching activities: indeed, within the meta-didactic transposition, practices become meta-didactic practices as they refer to the activities and reflections that characterise teacher education processes. Meta-didactic practices refer to the practices and theoretical reflections developed within a community of inquiry in the sense of Jaworski, as mentioned above.

4. Methodology

In the initial phase of the research, the aim is to identify the training needs of in- and pre-service teachers. To this purpose, a questionnaire was designed and administered. It is declined in six distinct versions, formulated according to the group of teachers involved (pre- and in-service teachers) and the school level in which they are teaching or they will be teaching (focusing on grade 5, 8 and 10). The questionnaire includes questions based on specific INVALSI items selected from the GESTINV database (www.gestinv.it). The items were chosen as representative examples of macro phenomena at national level: multiple-choice items with a notably low percentage of correct responses among students and whose distractors (incorrect answers) can be framed with mathematics education constructs known in literature. Additionally, the questionnaire includes "transversal" questions that are not directly related to the proposed INVALSI items. These questions are specifically designed to explore participants' perceptions of the purpose and usefulness of the INVALSI tests, as well as their impact on current or future teaching practices. Previous works by the INVALSI-Didactics Disciplinary Group, which is part of the observatory of the SIRD – Italian Society for Educational Research[1] (Arzarello and Ferretti, 2022; Ferretti *et al.*, 2022), have shown how a questionnaire using INVALSI items can provide useful information about in-service grade 5 teachers.

In this paper, we focus specifically on questions based on selected INVALSI items to explore in-service and pre-service teachers' anticipations and interpretations of the most common mistakes students make when answering these items. As we will see, Pedagogical Content Knowledge (PCK) (Carrillo-Yañez *et al.*, 2018) is involved in this activity, and the analysis provides insights into the needs of both in-service and pre-service teachers.

In the try-out phase of the questionnaire, we selected four INVALSI items for each grade (grades 5, 8, and 10), for a total of 12 INVALSI items.

5. Results

131 pre-service and in-service teachers participated in the try-out of the questionnaire, including 27 in-service and 47 pre-service teachers for grade 5, 12 in-service and 11 pre-service for grade 8, and 14 in-service and 20 pre-service for grade 10. The questionnaire was administered online, which made it possible to reach teachers from a variety of schools and universities across Italy, thus ensuring a heterogeneous set of respondents. While the sample was not designed to mirror the national distribution of Mathematics teachers, its diversity supports the exploratory aims of our study. In presenting the results, we refer in particular to the responses of secondary school

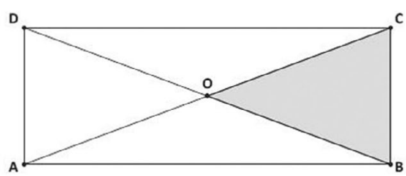
1 https://www.sird.it/en/home-en/.

teachers, as the overall pattern of answers was comparable between in-service and pre-service teachers; however, similar tendencies were also found among primary teachers.

We now present some of the results we have collected and analysed with regard to the interpretation of mistakes, i.e. what we have called "distractor analysis". For each INVALSI item included in the questionnaire, we asked teachers to answer the following question (without revealing the national percentage of correct answers): "The correct option is option X. What do you think was the most popular wrong option at national level? Why?". In this way, we can investigate whether teachers are aware of the mistakes students are most likely to make and, simultaneously, whether they have a clear understanding of the underlying causes of those mistakes.

An example is item 6 of the 2012 INVALSI test for grade 8 students. It asks whether it is possible to calculate the area of the triangle (in grey) knowing the area of the rectangle (Fig. 2).

The figure shows rectangle ABCD with its diagonals. If you know the area of the rectangle, can you calculate the area of the shaded triangle?



A. ☐ No, because the four triangles with vertex O are not all congruent.

B. ☐ No, because I don't know the dimensions of the rectangle.

C. ☐ Yes, because the four triangles with vertex O are equivalent.

D. ☐ Yes, because the four triangles with vertex O are isosceles.

Fig. 2 – Item 6 of 2012 INVALSI test for grade 8 (translation by the authors)

The data from the GESTINV database (Fig. 3) show that only 24.1% of the sample selected the correct answer C, while the most frequently chosen distractor B was selected by 30.8% of the students. This option suggests that the area of the triangle cannot be calculated due to the unknown dimensions of the rectangle. Subsequently, 26.9% of the respondents selected A (the area cannot be known because the triangles are not equivalent) and 16.4% chose D (stating that it can be known because the triangles are isosceles). Examining the characteristic curves of the item, it is evident that distractor B is selected by students across all ability levels with a similar probability. A similar pattern is observed for distractor A, which is chosen by a percentage almost identical to that of distractor B. It can also be observed that the characteristic curve for the correct answer does not closely align with the ideal curve: there is an underestimation for students with lower ability in the test and an overestimation for higher ones.

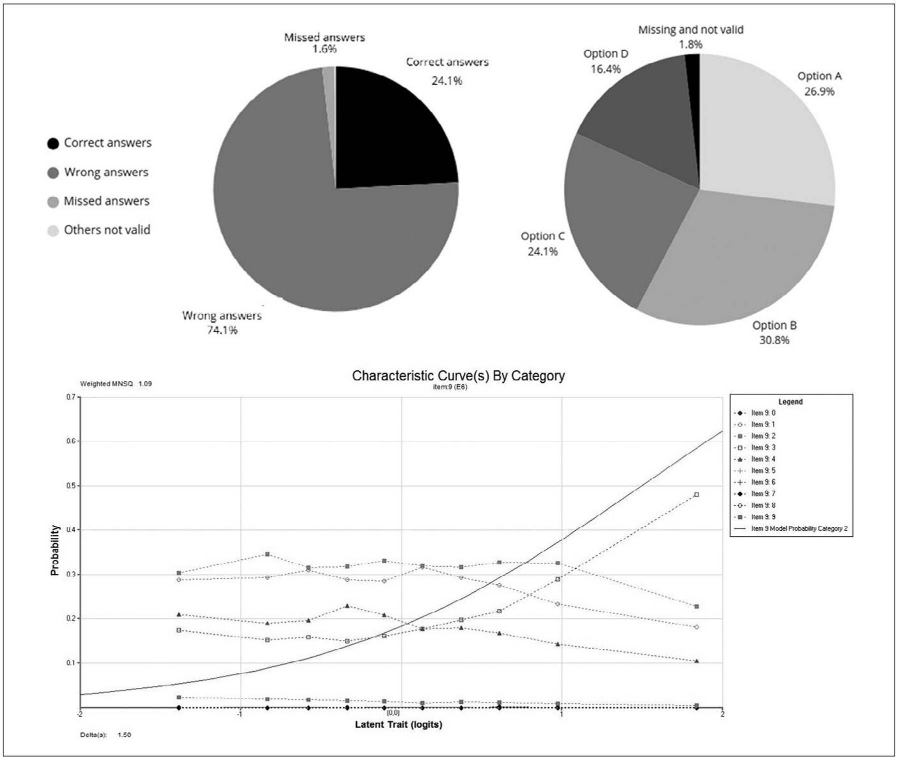


Fig. 3 – National results for item INVALSI 6 of the 2012 for grade 8 students

At the top left, the national percentages of correct, incorrect, and invalid responses; at the top right, the national percentages for each answer option; below, the characteristic curve describing the pattern of distractor choices at the national level in relation to students' ability levels across the entire test

Table 1 represents the analysis of the low secondary school teachers' answers. More than half of in-services identify as the most likely distractor the one that is actually chosen by the national sample, even though a good portion (33.3%) states that the most chosen wrong answer by students might be option A. Conversely, among pre-service teachers, almost 73% claim that the most chosen distractor is option A and only 18% align with the national results.

Tab. 1 – The main choices and reasons of lower secondary school teachers (12 in-service and 11 pre-service) who responded to the questionnaire, about the distractor analysis of item 6 of 2012 INVALSI test for grade 8

In-service teachers (12)	*Pre-service teachers (11)*
58.3% choose option B, because *the absence of numerical data prevents calculations leading to the result* (e.g. "B because they are looking for the values of the dimensions to calculate the area but do not know them")	18% choose option B because of the *absence of numerical data* (e. g. "B because pupils expect dimensions as they are always given in their textbooks and problems")
33.3% choose option A because *visually the triangles are different* (e. g. "A because it stops at a glance")	72.2% choose option A because *visually the triangles are different* (e. g. "Maybe A because to the eye the triangles do not look equivalent") or because of the presence of the *"equal" distractor element* (e. g. "The most frequently chosen wrong option was A, because the word "equal" leads the student to the mistake")

Despite this difference between in-service and pre-service, the reasons given by both groups to explain their choice of distractors are the same. As it can be seen in Table 1, both state that the choice of distractor B could be due to the absence of numerical data in the problem text and, in a good percentage, they also give the same interpretation for the choice of option A (i.e. that students perceive the triangles to be visually different).

The awareness of pupils' mistakes can be interpreted as evidence of Knowledge of Features of Learning of Mathematics (KFLM), a Pedagogical Content Knowledge subdomain of the MTSK model. It includes knowing how students might act, their mistakes and difficulties in specific topics, and, in general, how students interact with Mathematics while also considering affective aspects. For example, comment like "B because pupils expect dimensions as they are always given in their textbooks and problems" can be considered as Knowledge of Mathematics Teaching (KMT), understood as the way a topic is typically taught.

It appears that in-service teachers' anticipation of students' mistakes is more aligned with the INVALSI results than that of pre-service teachers. In this case, it can be regarded as a deeper KFLM, which appears to be more developed in in-service teachers than in pre-service teachers. However, this does not also apply to the interpretation because, as it can be seen, teachers (both pre- and in-service) that choose respectively the two wrong options A and B determine the same reason for both distractors.

Another example is from the questionnaire for high secondary school teachers. Item 21 of the 2017 INVALSI tests for grade 10 students (Fig. 4) is presented to the teachers; it asks to calculate the slope of a straight line, knowing two points belonging to it.

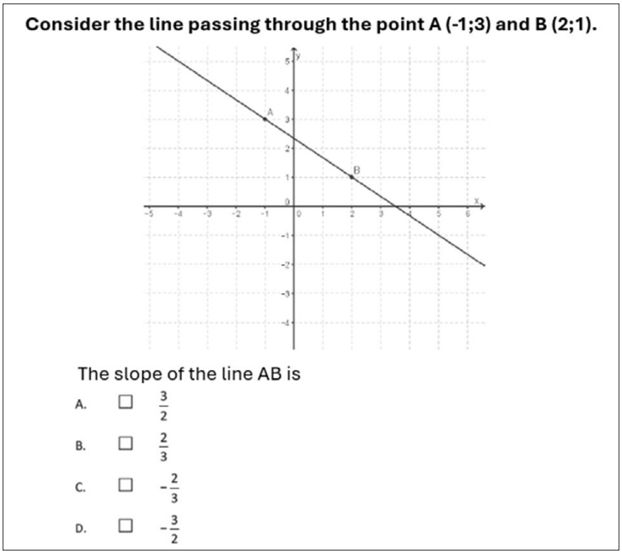


Fig. 4 – Item 21 of 2017 INVALSI test for grade 10 (translation by the authors)

The correct answer is chosen by 29.6% of the students nationwide and the percentages of distractor choices are very similar to each other, especially for the two most chosen: D is selected by 24.1% and A by 23.7%. This can also be observed from the characteristic curves, which are much “closer” to each other than the distractor curves of the item shown in the previous example. As before, however, the ideal curve underestimates students with lower ability in the test and overestimates higher ones (Fig. 5).

In this instance, most in-service teachers do not interpret mistakes in line with the national results: in fact, only 21% choose option D and even only 7% select option A. On the other hand, the percentages of pre-service teachers are more “Equi distributed”, as indeed are the students’ answers, even though they do not correspond to the national sample; indeed, we recall that, in order, the most chosen distractor was option D, followed by option A and then B. Pre-services selected mainly D (35% of the respondent), then B (25%) and only lastly A (20%). It is also notable that the motivations given by the pre-servicers are also very varied, compared to those of the in-service (Tab. 2). The latter, in fact, interpret the mistake, regardless of the distractor, as a problem in the use of the formula for calculating the angular coefficient; this is also mentioned by the pre-service, but other reasons (such as confusion between the angular coefficient and the ordinate at the origin, influence of the data in the request, mention of the cathexes of the right triangle, etc.) are taken into account. Hence, in this case, it seems that pre-service teachers are more conscious of possible students' mistakes, and they also give many different interpretations, which are not considered by in-service teachers.

Similarly to the previous case, the responses of high secondary school teachers can also be classified as evidence of aspects of KFLM. In particular, it is observed that, regardless of the selected distractor, most teachers attribute the error to a difficulty related to students' knowledge and/or application of the formula. These reflections can also be interpreted as elements of the subdomain of KMT. It is plausible that teachers refer to established teaching practices, which represent habitual strategies for addressing the topic in class (for example, specifically through the introduction of the formula). Consequently, they may be more inclined to attribute the error to a critical issue related to this teaching approach. In this sense, pre-service teachers may have proposed more diverse interpretations, as they have not yet internalized such practices, given that they do not have direct classroom teaching experience.

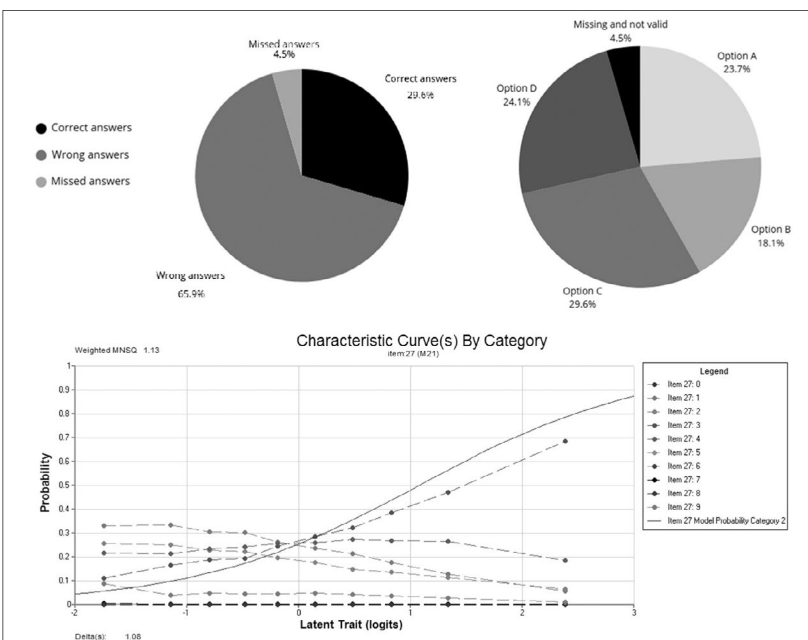


Fig. 5 – National results for item INVALSI 21 of the 2017 for grade 10 students

At the top left, the national percentages of correct, incorrect, and invalid responses; at the top right, the national percentages for each answer option; below, the characteristic curve

Tab. 2 – The main choices and reasons of high secondary school teachers (14 in-service and 20 pre-service) who responded to the questionnaire, about the distractor analysis of item 21 of 2017 INVALSI test for grade 10

In-service teachers (14)	*Pre-service teachers (20)*
21% choose option D because of *problems with the formula* (e.g. "D, lack of knowledge of the correct formula for calculating the slope of the line")	35% choose option D because of *problems with the formula* (e.g. "D because they invert delta y and delta x") or because they *are influenced by the data* (e.g. "D because they read the coordinates of the points from the graph instead of differentiating them")
7% choose option A because of *problems with the formula* (e.g. "A because students confuse x with y and confuse the use of signs")	20% choose option A because they *are influenced by the data* (e.g. "A because 3 and 2 are also two numbers that appear in the data points (3 first and 2 second)"), or they *incorrectly identify the cathexes of the triangle* (e.g. "I assume answer A because, when considering the points on the plane, they could observe the lengths of the cathexes formed by the line with the axes from the abscissas"), or *they confuse the angular coefficient and the ordinate at the origin* (e.g. "A, because the line intersects the y-axis in 3/2").
57% choose option B because they correctly *apply the formula but do not take into account the sign* (e.g. "In my opinion, the most popular option is B. They "correctly" apply the formula for the angular coeff. but do not report the – sign (which is moreover in the denominator, something students are not used to)") or because *they do not reason on the slope* (e. g. "B, because the student does not take the negative slope into account")	25% choose option B because of the *omission of the minus sign* (e.g. "B because signs are often considered irrelevant")

5. Conclusion

The INVALSI tests serve as a valuable tool for examining the knowledge of Mathematics teachers. By leveraging data from the national sample, these tests make it possible to guide both pre-service and in-service teachers in reflecting on the potential mistakes made by students and on their possible underlying causes. The present study adopted this perspective and analysed how teachers interpreted selected INVALSI items that present frequent student errors. The examples discussed are only a subset of those included in the questionnaire and are intended as illustrative of the types of knowledge and beliefs that emerge when teachers engage with large-scale assessment data through the lens of the MTSK model. Similar patterns were observed in teachers' responses to other items, which could not be included for reasons of space but point in the same direction.

Overall, the findings suggest that much of the knowledge mobilised during the interpretation process can be framed within specific subdomains of Pedagogical Content Knowledge, particularly KFLM and KMT. More broadly, the study shows how the MTSK framework can serve as a useful theoretical lens for characterising teachers' reasoning when analysing students' performance in standardised assessments. At the same time, the results highlight a potential challenge in teachers' perceptions, as participants often experienced difficulties in identifying the most frequent mistakes made by students at the national level.

Differences also emerged between the interpretations offered by in-service and pre-service teachers, as had already been observed in other studies conducted within the MaSt project (for further details, see Cibien, 2025). Although the qualitative nature of the data does not allow us to determine whether these differences are statistically significant, they seem to indicate that classroom experience may influence the ways in which teachers attribute causes to students' errors and the knowledge domains they activate. Conversely, pre-service teachers sometimes proposed a broader range of possible explanations, which may suggest a certain "flexibility" in their approach. This observation, however, warrants further investigation through a more structured quantitative comparison.

On this basis, these insights contribute to identifying strengths and weaknesses in teachers' processes of recognising and interpreting students' mistakes. This study has allowed us to gather significant information to be used as a foundation for the implementation of the model for professional development for teachers addressed by the main Project MaSt. Teachers' difficulties in identifying and interpreting student errors must be taken into consideration when designing the training model, meaning that it would be beneficial to implement activities aimed at fostering this awareness.

A limitation of the study is the small size of our sample. In the new phase of the research, efforts are being made to reach a larger and more diverse sample of in-service and pre-service teachers. In this broader context, a quantitative study will be conducted to examine in greater depth the educational needs and beliefs of Italian Mathematics teachers regarding the INVALSI tests and to verify whether the tendencies observed in the present qualitative analysis are confirmed at scale.

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4. Co-disciplinarity to reduce early leaving school: Math education in an innovative and integrative dialogue with other disciplines

by Antonella Montone, Michele Giuliano Fiorentino

This research explores early school leaving in Italy and Europe, particularly in lower and upper secondary education. A key issue is students' loss of meaning and perceived irrelevance of Mathematics in everyday life. This phenomenon is especially pronounced in vocational schools, where subject-specific learning is closely tied to future professional skills.

The study aims to enhance the meaning of Mathematics by connecting it to other disciplines and, conversely, using real-world problems from various fields to highlight the necessity of Mathematics. To achieve this, co-disciplinary educational pathways are designed, fostering greater student engagement, especially among those at risk of dropping out.

A valuable framework for this approach is the Theory of Semiotic Mediation (Bartolini Bussi and Mariotti, 2008), which examines students' interactions with artifacts from different disciplines. These interactions help develop both mathematical understanding and domain-specific knowledge.

This interdisciplinary perspective aligns with the co-disciplinary approach (Blanchard-Laville, 2000), which appears to be particularly effective. Preliminary analyses of classroom experiments suggest that this method enhances the development of mathematical concepts across disciplines while also increasing active participation, particularly among at-risk students.

Questa ricerca esplora l'abbandono scolastico precoce in Italia e in Europa, in particolare nell'istruzione secondaria inferiore e superiore. Una questione centrale è la perdita di significato da parte degli studenti e la percezione dell'irrilevanza della Matematica nella vita quotidiana. Questo fenomeno è particolarmente pronunciato nelle scuole professionali, dove l'apprendimento delle materie è strettamente legato alle competenze professionali future.

Lo studio mira a rafforzare il significato della Matematica collegandola ad altre discipline e, viceversa, utilizzando problemi reali provenienti da

diversi ambiti per evidenziare la necessità della Matematica. Per raggiungere questo obiettivo, vengono progettati percorsi educativi co-disciplinari, favorendo un maggiore coinvolgimento degli studenti, soprattutto di quelli a rischio di abbandono scolastico.

Un quadro teorico prezioso per questo approccio è la Teoria della mediazione semiotica (Bartolini Bussi e Mariotti, 2008), che analizza le interazioni degli studenti con artefatti provenienti da diverse discipline. Queste interazioni aiutano a sviluppare sia la comprensione Matematica sia la conoscenza specifica di ciascun ambito.

Questa prospettiva interdisciplinare si allinea con l'approccio co-disciplinare (Blanchard-Laville, 2000), che sembra essere particolarmente efficace. Le analisi preliminari degli esperimenti in aula suggeriscono che questo metodo favorisce lo sviluppo dei concetti matematici attraverso le discipline, aumentando al contempo la partecipazione attiva, soprattutto tra gli studenti a rischio.

1. Introduction

This research aims to address school dropout in vocational schools by designing co-disciplinary training courses, co-developed and co-designed by Mathematics and vocational subject teachers. The goal is to highlight the existing connections between these subjects, making Mathematics more meaningful (Wake, 2014) in relation to vocational disciplines and, conversely, giving vocational subjects deeper significance through Mathematics.

The study seeks to demonstrate that integrating disciplinary content with vocational training is essential for recognizing the relevance of subjects traditionally seen as unrelated to professional fields – such as Mathematics and Italian – while also enhancing vocational skills and career development. Many vocational students, particularly in their first two years, struggle academically, challenging the institutional training framework. This issue is further emphasized by contradictions in national education guidelines (MIM, 2019).

To address this, an innovative teacher training program is needed, as Mathematics teachers often lack awareness of the specific mathematical concepts relevant to vocational subjects, while vocational teachers require mathematical training to effectively teach certain aspects of their field.

The research is grounded in the Theory of Semiotic Mediation (TSM) and further enriched by the concept of Boundary Object Artefacts (BOA), a specially designed tools that facilitate the integration of Mathematics and vocational disciplines. This article analyses the semiotic potential of these artefacts and presents preliminary results from two co-disciplinary interventions in Mathematics and Fashion and Mathematics and Chemistry. Through students' perspectives, early findings suggest that a co-disciplinary approach can play a significant role in addressing school dropout.

2. Theoretical framework

Drawing from the growing understanding of mathematical activities in work contexts, it is essential to gain new insights into the role of Mathematics, both as an academic discipline and as a versatile tool applied across various aspects of human life (Wake, 2014). Learning in these contexts can be mediated through specially designed Boundary Object Artefacts (BOAs), inspired by Akkerman and Bakker' s (2011) concept of boundary objects. Our interpretation of BOAs serves to coordinate different disciplines, maintaining their relevance within each field while remaining adaptable for interdisciplinary use (Star, 1995).

Within the Theory of Semiotic Mediation (TSM), we define a BOA as any human-created artefact designed for a specific purpose. Our perspective expands on the traditional TSM notion of artefacts, as BOAs not only evoke mathematical concepts but also convey vocational meanings. This aligns with Rabardel's (1995) concept of utilization schemes and the core idea of semiotic potential in TSM, the dual relationship between an artefact's personal meaning, the mathematical concepts it evokes, and its vocational applications.

BOAs function as operational tools for decision-making and instructional design, acting as "reference situations" that shape conceptual understanding (Vergnaud, 1994). Identifying effective BOAs requires collaboration among experts from different disciplines, working toward a shared objective. This process follows a co-learning inquiry approach, where teachers and researchers engage in joint exploration, using inquiry as a mediating tool. As Wagner (1997) states, in co-learning agreements, both researchers and practitioners actively participate in educational processes, learning from one another while deepening their own understanding of their respective fields and their connections to educational institutions.

A key objective of this approach is for researchers and teachers to collaboratively explore and improve the teaching and learning of Mathematics. By incorporating BOAs from vocational contexts, we aim to deepen mathematical understanding while simultaneously enriching vocational learning. The design, analysis, and implementation of BOAs follow the TSM framework (Bartolini Bussi and Mariotti, 2008), which is rooted in Vygotskian socio-constructivism. The use of artefacts and the deployment of their semiotic potential leads students to produce personal knowledge, which through mathematical discussions, orchestrated by the teacher, is gradually transformed into shared knowledge. The notion of semiotic potential expresses the relationship between the personal meanings emerging from the experience of acting with the artefact and the mathematical meanings recognizable by the expert in such actions; its strict dependence on the task, which must be carried out by the students, makes the artefact a key tool for designing appropriate tasks. The TSM was used both by teachers during the co-planning of the training courses and in the analysis of the semiotic potential of the artefacts, and by researchers to analyse the results of the experimental activity in reference to Mathematics and the vocational discipline. Specifically, in the research proposed here, the tasks are appropriately designed to ensure that during the use of BOAs they can bring out their semiotic potential in the same way both in relation to Mathematics and in relation to the vocational discipline involved. Their use and their dual semiotic potential is then made explicit in collective discussions (Bartolini Bussi, 1998) orchestrated by the teachers. In collective discussions it is also fundamental to analyse language, which especially in Mathematics sees the use of mathematical registers as extreme forms of evolved registers (Ferrari, 2021).

The interactions between different disciplines, let emerge different approaches from the literature in relation to the possible interactions between them, on whose definition there is general agreement: multidisciplinarity, in which the relationship between the disciplines consists in the segmentation of the work without a real interaction (Stokols *et al.*, 2010); trans-disciplinarity, in which the interaction between disciplines leads to the establishment of a new discipline with its own epistemology (Aboelela *et al.*, 2007); interdisciplinarity, in which the encounter between disciplines takes place through the integration of some aspects of the research (content, theory, methods). Any interdisciplinary intention requires and implies some overcoming of disciplinary boundaries, i.e. the entry by researchers into unfamiliar fields of study (Choi, 2017). For this study, it seems to be necessary to go beyond these three approaches and sustain the co-disciplinary approach. According to Blanchard-Laville (2000), co-disciplinarity involves researchers collaboratively engaging in a shared project, fostering "co-thinking" rather than simply aligning perspectives. This dynamic process encourages the emergence of new ideas and solutions. In our study, interdisciplinary collaboration begins with joint planning and extends through observation, monitoring, and reflection on the training pathways. A co-disciplinary approach is particularly suitable here, as it facilitates meaningful dialogue and cooperation among teachers from different fields.

3. The research methodology and experimental design

This paper focuses on a specific experimental educational path of a general project that is design-based research (Swan, 2020). In particular, the researchers, authors of this study, designed operational phases in which the solution of a problem represents the space where the co-disciplinarity between professional subject and Mathematics subject takes place by the means of a BOA.

The overall objective of the study is to establish meaningful connections between Mathematics and vocational subjects, ensuring that each discipline enhances the understanding of the other. Our central assumption is that mathematical knowledge plays a crucial role in meeting professional requirements (Winther, 2016). Aligned with the perspective of Lave and Wenger-Trayner (2008), we assert that fundamental skills such as Mathematics provide a foundation for the development of vocational competencies, a concept rooted in the theory of situated learning.

The specific aim of this research is to explore how co-disciplinary educational pathways, co-designed and co-taught by both Mathematics and vocational teachers, can highlight the intrinsic link between vocational training and Mathematics. This approach particularly focuses on engaging students at risk of dropping out of school.

We hypothesize that this interdisciplinary approach enables students to find meaning in both Mathematics and vocational theories relevant to their future professions. Furthermore, we propose that enhancing learning in these disciplines increases students' motivation and reinforces the perceived usefulness of school education in their career development. Consequently, we pose the following research questions:

- Does a co-disciplinary approach utilizing BOAs facilitate the evolution of signs and enhance learning?
- Can student engagement in such structured activities, along with their learning progression, motivate them to recognize the value of school in their professional development?

The teaching intervention was organized into several preparatory design phases before classroom implementation:

- Phase 1: implementation of co-disciplinary work. Mathematics and vocational teachers discussed the potential of integrating their subjects to create a synergistic learning experience.
- Phase 2: identification of BOAs. Teachers selected BOAs commonly used in vocational contexts and analysed their semiotic potential from both mathematical and vocational perspectives. The chosen BOAs included the paper model, the quartometer, and a flask containing colored balls.
- Phase 3: structuring educational pathways. The analysis of the "dual" semiotic potential of the BOAs, linking mathematical concepts with vocational applications, facilitated the design of a co-disciplinary teaching intervention (Fiorentino *et al.*, 2021, Fiorentino *et al.*, 2023).
- Phase 4: educational intervention. The study involved two classes of approximately 20 students, implementing two co-disciplinary teaching pathways: one integrating Mathematics with fashion and the other with Chemistry. The lessons began with a problem-based scenario involving BOAs, followed by collective discussions, additional problem-solving exercises to reinforce mathematical concepts, and a concluding discussion. The Mathematics and vocational teachers co-designed and co-taught the activities in collaboration with researchers, who observed, recorded, and analysed the sessions.

In both classes, five to seven students were at risk of dropping out and had consistently low grades in both subjects. Additionally, the schools recorded a 70% rate of unjustified absences over three months. To address the research aim, we analysed transcripts of the Mathematical Discussions related to problem-solving and BOA usage. To validate the teaching pathways, we conducted a qualitative analysis based on the criteria of credibility, reliability, transferability, and confirmability (Guba, 1981).

This study is part of a broader project involving five vocational institutes, 15 in-service teachers, and approximately 300 students in the first two years of vocational education. In Italy, vocational schools (grade 9 to grade 13) prepare students for both academic studies and professional careers. The experiments discussed here were conducted in grade 10 (age 15), where dropout rates in the first two years often reach 25%.

The study adopts an exploratory qualitative perspective. Given the limited number of participants involved in the experimental phases, the findings should be interpreted as preliminary evidence, aimed at generating hypotheses rather than producing generalizable results.

In the following sections, we will analyse these experiments in detail.

4. “Vocational” artefacts: the case of the paper pattern and the quartometer

The experiment on Fashion and Mathematics took place in a grade 10 class specializing in Industry and Crafts for Made in Italy at a vocational school in southern Italy. Prior to the activity, the Fashion teacher had introduced students to the use of paper patterns (see Fig. 1) for trouser production, while the Mathematics teacher had covered concepts of proportion and symmetry.

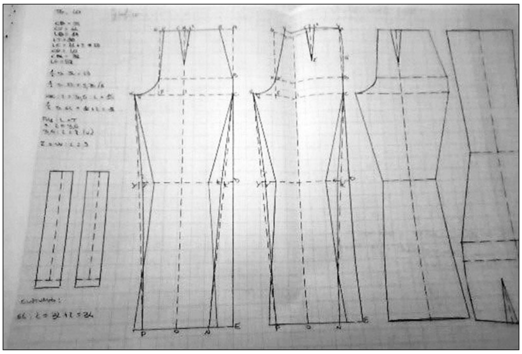

Fig. 1 – The paper pattern of a trouser

In the first activity, students were presented with the task: “Reproduce the pattern for making a bodice on a paper pattern”. Working in pairs, they were provided with an artefact to complete the task. A collective discussion, led by both teachers, followed to highlight the mathematical concepts of proportion and symmetry and their vocational applications.

The artefact used facilitates the modelling and creation of garments by incorporating algebraic and geometric relationships, offering a flat representation of the garment before its actual construction. The selection of a specific garment section introduces the mathematical idea of the axis of symmetry, enabling the exact reproduction of a quarter of the model. From a vocational perspective, this corresponds to the concept of fabric folding. Additionally, identifying positions in relation to one another on the pattern connects to the mathematical principle of proportionality, paralleling the vocational understanding of the proportional relationship between waist and hip circumferences. When transitioning between different sizes, students engage with mathematical proportionality, mirroring the vocational practice of adjusting patterns for size variations.

The second phase of the activity required students to scale up the garment design from a 1:4 model to its full-size version using a specialized artefact: the “quartometer” (see Fig. 2). This tool, a ruler designed with a 1:4 ratio, is essential due to its alignment with the body’s two axial symmetries. For instance, in trouser design, the axis running through the center of the crotch and the lateral axis dividing the front from the back are crucial considerations. The 1:4 ratio ensures that all trouser measurements remain proportional, resulting in a precisely scaled model.

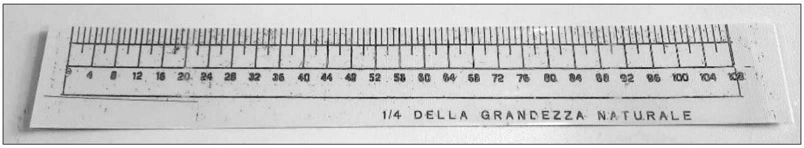

Fig. 2 – The “quartometer”

5. “Vocational” artefacts: the case of the flask and the balls

The experiment involving Chemistry and Mathematics was conducted in a second-year class of the dental technician program at IISS “Sergio Cosmai” in Bisceglie. The Chemistry teacher had already covered the concept of chemical solutions, while the Mathematics teacher had introduced the concept of percentage.

In the first activity, students were asked to solve the following problem: “Simulate a 100 ml solution with a 50% concentration”. The artefact used (see Fig. 3) consisted of a transparent flask and balls of different colors (red and white). The students were provided with the artifact and worked in groups of five.

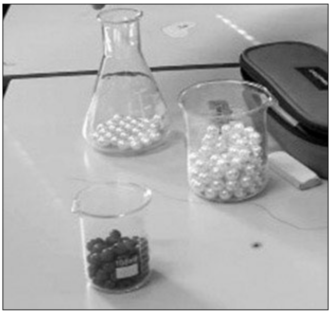

Fig 3 – Transparent flasks and coloured balls

At the end of the activity, both teachers led a collective discussion with the goal of highlighting the mathematical concept of percentage in relation to the chemical concept of solution concentration.

In the simulation, the total number of balls used represented the quantity of solution in millilitres, with each ball corresponding to 1 ml. The red balls represented the solute, while the white balls represented the solvent.

The choice of two different colours for the balls served a dual purpose: from a chemical perspective, it illustrated the difference between the molecules of two different substances; from a mathematical perspective, it conveyed the idea of the ratio between two different quantities.

Placing the balls in a flask allowed for subsequent mixing, which evoked: from a chemical standpoint, the process of mixing two substances; from a mathematical standpoint, the concept of a rational number representing a percentage.

6. Data collection and analysis criteria

The analysis concerns the co-disciplinary discussion carried out in the following phase 4 to group work in which different problem solutions emerged. According to the specific hypothesis of this study, the following analysis attempts to highlight how the co-disciplinary training path has given meaning to the mathematical concepts compared to the concepts of the vocational disciplines and vice versa gives meaning to the vocational concept through the mathematical concepts in play.

6.1. The case of Mathematics and Fashion

The analysis of some extracts from the experimentation concerning Mathematics and fashion is given. In particular, here, the vocational discipline teacher suggests making a bodice with the help of the BOA pattern. In doing this the students begin to draw one of the four parts of the bodice and place marks in particular positions. At this point the Mathematics teacher intervenes by starting the collective discussion:

Mathematics teacher: “What are you tracking?”.

Giulia: “The flank line”.

Mathematics teacher: “And why are you drawing it right there?”.

Giulia: “It is central, there is a calculation to define its position... from F we find G, then divided by two and we get the position… it is not central to everything, but only to this point F. This straight line represents the front and back of the dress…”.

Fashion teacher: "But are you only doing one side?".

Vanessa: "Yes, I'm only doing the back side, because there is a front part and a back part... but then there is a "roundabout" here... and then this part overlaps because they are symmetrical. Then the different elements are added, here is the shoulder and here is the neck (point with their finger to the lines on the paper pattern relating to the parts of the dress that correspond to the parts of the body)".

Marzia: "They are obviously not the same, but there are some symmetrical parts, once they overlap the body from here to here... (and indicates the position on your body of where the dress will be placed)... it is the half circumference bust... divided by the waist circumference and with this calculation we have the recovery... clearly based on the sizes... in 52, it is half of 104... 42 is half of 84 which is valid for size 50. Making (52-42): 4... because there are 4 thoughts, because we have four thoughts. It comes out 2.5. This 2.5 represents the gap".

Mathematics teacher: "Why the semi-circumference?".

Marzia: "It's done half the circumference why we are doing it half of the body... because when we let's go and place (the paper pattern) we place it double… because this is one half of the bodice".

Arianna: "Why it is reproduced on the double fabric... because the two parts are one front and one rear and then they come together... so the dress is divided into four parts... it is placed on the double fabric, that is, a large folded fabric... We only make one, because when it is then placed on the fabric two parts come out...".

In this extract, it seems clear how much the BOA paper model manages to make students dialogue both by providing justifications relating to Mathematics and to the vocational discipline. In the discussion, in fact, in the underlined parts there are explicit references to the mathematical meanings of "symmetrical parts" and circumference, just as the references to the meanings of the discipline relating to what they are representing represents in the body appear to be clear. The discussion continues and the teacher, with the aim of moving on to the second BOA, the quartometer, to bring out the concepts of ratio and proportions asks:

Mathematics teacher: "How do I go from a small drawing to having a natural drawing?".

Giulia: "Because we make the small drawing which is a quarter of the natural size... we have a particular ruler... for example, 1 cm stands for 4 cm… every centimeter corresponds to 4. Then we use a table with the measurements relating to the sizes… the pattern respects the proportions, everything is magnified 4 times… ¼ is a fundamental fraction for making clothes... starting from the drawing on a small sheet of paper, to move on to the creation of the actual size paper pattern, to finish with the cutting of the fabric and the composition of the dress with these wing the 4 symmetrical parts".

Arianna: "I finally understand why we use the symbol ¼ for the ratio! It seems to me like something is getting clearer…".

Marzia: "I agree with Arianna, by really understanding what I'm doing, I prefer to stay here in class, so that I could better work with paper pattern on my own".

In the response reported here, the reference made to the mathematical meaning of the 1:4 reduction scale with which the BOA quartometer is made seems evident, indeed Giulia expressly says: "1 cm stands for 4 cm... every centimetre corresponds to 4". Furthermore, the non-randomness of the ¼ ratio also emerges, as it is expressly stated that "¼ is a fundamental fraction for the creation of clothes", referring to the "sewing of the 4… symmetrical parts". Moreover, Arianna highlights how the signs used are useful for her personal understanding of ratio concept. This seems giving an additional value to the Mathematics learning.

Finally in the last sentences by Marzia it seems to us evident how the involvement of students in such activities, could improve their learning and, above all, believe in the usefulness of school in their personal professional development.

The analysis of the excerpts related to Mathematics and Fashion highlights how the use of the paper pattern and the quartometer as Boundary Object Artefacts supports the emergence of mathematical meanings closely intertwined with vocational knowledge. Students' utterances show a progressive evolution of signs, in which mathematical concepts such as symmetry, proportion and ratio are mobilised to interpret vocational actions and, at the same time, vocational practices provide concrete meaning to abstract mathematical notions.

Particularly relevant is the active participation of students identified as being at risk of early school leaving. Their interventions reveal not only conceptual understanding, but also an increased engagement in the activity and a growing awareness of the usefulness of mathematical knowledge for their future professional practices. From a semiotic mediation perspective, the BOAs appear to function as effective mediators, fostering sense-making processes that bridge Mathematics and vocational learning within a co-disciplinary framework.

6.2. The case of Mathematics and Chemistry

The analysis concerns the co-disciplinary discussion carried out in phase 5 after the group work in which different problem's solutions emerged. According to the specific hypothesis of this study, the following analysis attempts to highlight how this co-disciplinary educational path makes sense of

Mathematics concept of percentage with respect to the chemistry concept of solution and vice-versa gives meaning to the concept of solution concentration through the Mathematics concept at stake.

Mathematics teacher: "Does anyone want to tell me what you did?".

Morena: "We were asked to prepare a 100 ml solution concentrated at 50%. We had the white balls standing for solvent and the red balls standing for solute and an Erlenmeyer flask. We filled the flask with 50 red balls of solute and 50 white balls of solvent".

Mathematics teacher: "Therefore, you have all created a solution of 100 ml at 50%. Why is Enxy stirring?".

Francesca: "At the beginning, when we put the balls in the flask they were concentrated by colour and then we mixed them up to obtain the solution".

Chemistry teacher: "And how were the solutions created?".

Enxy: "We counted the quantities of the molecules...".

Morena: "We took 50-50 because the concentration the problem asked for, was 50%, that is ½".

In this excerpt, it seems to us that the students give meaning to the mathematical concept of 50% through the action of putting half balls of one colour and half of the other into the flask. As expected from a chemical point of view the ball evokes the idea of a molecule, as Enxy says, and the different colours highlight the two substances, solvent and solute. From a mathematical point of view, the possibility of having discrete objects such as balls makes it possible to realise the percentage and to establish how many balls of one colour and how many of the other you need to put in the flask to obtain a solution of 100 ml concentrated at 50%. The BOA seems to bring together the chemical concept of solution and the mathematical sense of percentage that make it possible to realise the solution requested.

The discussion proceeds and the Mathematics teacher proposes a new problem linked to chemical concentration with the aim of bringing out the mathematical meaning of percentage for any reference quantity:

Mathematics teacher: "Enxy says we made a 50% concentration of 100 ml because each ball represents 1 ml. If I asked you to make a 20% solution of 100 ml, what would you do?".

Francesca: "20 and 20".

Morena: "Yes, 20 white balls and 20 red balls".

Francesca: "But by making 20 red balls and 20 white balls we get 40 balls and not 100 we have 20% of 100ml…".

In this excerpt, Francesca and Morena give a wrong solution, probably because they mechanically transfer the solution of the previous problem to the new one, highlighting a lack of sense. Francesca puts in discussion her solution because she observes the BOA and she finds something wrong. She did not think it through enough to give a correct solution. The following interventions of Nicola and Ginevra help Francesca to re-think her solution.

Nicola: "Something doesn't add up... 20 balls of solute and 20 balls of solvent are always 50% concentrated. Whereas at 20% I expect to find far fewer balls of solute... At 20% the solution is less concentrated, i.e. there should be far more white balls".

Ginevra: "Yes, I agree with Nicola, otherwise there would be no difference between concentration at 50% and 20%... But how many balls do we have to put in solute and how many in solvent?".

Francesca: "Ah yes, I got it! We must do 20 out of 100 and 80 out of 100…".

Carlotta: "Right! The solute is 20... Then 20 out of 100, so to get 100ml total, the solvent must be 80 balls. In other words, we did the ratio of 20 to 100, that is ⅕ of 100 balls. So, 20 red balls for the solute and ⅘ of 100 balls for the solvent, i.e. 80 white balls".

It seems to us that Nicola's statement reveals that the BOA is playing a fundamental role because it puts in evidence the mistake. Nicola draws a relationship between the balls, the number of the balls, as molecules, and the percentage. In the following words, he draws a relationship between the percentage (20%) and what he expected to find. The number of balls does not correspond to the ratio 20/100. Ginevra, in agreement with Nicola, goes further. Indeed, she shifted from what is evident by means of the BOA to the mathematical level, comparing the percentages 20% and 50%. The discussion based on the artefact's use linked to mathematical and chemical meanings, seems to lead Francesca to re-construct her solution. Indeed, her statement "we have to do 20 out of 100 and 80 out of 100" shows the evolution of her previous wrong idea. Carlotta offers a synthesis confirming what Francesca said. The use of the artefact from the TSM, permits Ginevra, Francesca and Carlotta (these students have been declared at risk of early school leaving by the Class Council) to intervene and give sense to what they are saying.

In the following excerpt, it emerges that students use percentages to determine the ratio of solute, solvent and solution in a given concentration.

Mathematics teacher: "If instead of 100ml, the solution was 50 ml at 20%, how would you calculate the number of balls?".

Francesca: "It halved…".

Enxy: "Both the solute and the solvent have halved".

Francesca: "The concentration, however, remains the same!".

Enxy: "Because they have the same quantities!".

Enxy: "Now let's put 10 balls of solute and 40 balls of solvent, as it was 20 and 80".
Chemistry teacher: "So, the concentration is what?".
Enxy: "It is the percentage of the balls of solute in comparison with the solution".
Mathematics teacher: "The percentage always remains the same, but what is this percentage? Before we talked about concentration, now we are talking about percentage, is there any connection?".
Morena: "The concentration used to be 50% of 100 ml, i.e. the ratio of the number of solute balls to the number of total balls is ½. Now the concentration is 20%, i.e. the ratio of the number of solute balls to the number of total balls is ⅕. Whether we made 100ml at 20% or 50ml at 20%, the percentage is the same, that is the ratio between solute and solution remains the same, even if the amount of solution changes".

Once again, the BOA used probably played the role of a boundary element between the two disciplines: the use of the artefact brings out the link between the disciplines and the specific concepts of each discipline, the relationship between the balls of different colours from a mathematical point of view represented by the percentage and from the point of view of chemistry represented the concept of concentration. In addition, during all the activity we could observe the presence and the interventions of the students most at risk of early school leaving. Moreover, all of them participated in all the activities, actively participating in the discussions. As we already stated, these students are often truancy at school and, even when they are present, don't participate actively in the lesson. This phenomenon leads us to think that, involving students in co-disciplinary activities aiming at making sense of disciplines' theoretical concepts, could help to reduce the early school leaving.

The analysis of the co-disciplinary discussion in the Mathematics and Chemistry pathway shows how the flask and coloured balls operate as Boundary Object Artefacts capable of making the mathematical concept of percentage meaningful through the chemical notion of solution concentration, and vice versa. The artefact allows students to reason about ratios, quantities and invariance of percentage across different reference amounts, supporting the evolution from initial misconceptions to more structured mathematical reasoning.

The discussion highlights how students, including those most at risk of early school leaving, actively engage in the collective construction of meaning by coordinating chemical and mathematical perspectives. The BOA plays a crucial role in making errors visible and negotiable, thus promoting reflective thinking and participation. These qualitative evidences suggest that co-disciplinary activities grounded in semiotic mediation can foster both conceptual understanding and engagement among students who typically show low participation in traditional classroom settings.

7. Preliminary results and concluding remarks

Throughout the activity it was possible to observe the presence and interventions of the students most at risk of early school leaving, such as Marzia and Giulia in the first case, and such as Francesca and Carlotta in the second one. Furthermore, all of them participated in the activities, actively intervening in the discussions. These students are often absent from school and, even when they are present, do not participate in class activities. Inspired by some intervention in the discussion, like the Marzia's one, we think that, by involving students in co-disciplinary activities aimed at giving meaning to the theoretical concepts of the disciplines, we can contribute to reducing school dropout. Students seem to demonstrate that they are rediscovering the meaning of symmetry when they encounter the difficulty to explain the position on their body. The students show that they rediscover a sense of percentage precisely when they encounter the difficulty of establishing the correct number of balls to achieve a 20% concentration. It is also evident, in students' synthesis offer, the reference to the mathematical concept to interpret the chemical concept and the chemical concept to explain the Mathematics concept of percentage. Furthermore, the reference to mathematical concepts to interpret the vocational concept itself and to the vocational concept to explain the mathematical concepts appears evident in the students' synthesis proposals. These connections seem to emerge thanks to the use of BOAs and allow students at risk of early leaving school to achieve the aims set by teachers. As expected, the experimentation of the co-designed and co-led educational path could be considered a good practice to reduce school dropout.

Although no causal relationship can be claimed, the observed increase in participation, sense-making and engagement among students at risk suggests that co-disciplinary pathways may represent a promising direction for preventing early school leaving.

Moreover, as highlighted throughout the analysis, the present study has some limitations. The number of students involved in the experimental activities is limited, no control group was considered, and data were collected over a short time span. For these reasons, the results cannot be generalized but should be interpreted as exploratory insights grounded in qualitative evidence.

Finally, future research could extend this study in several directions. First, the co-disciplinary approach could be tested on larger samples and in different vocational contexts. Second, the inclusion of control groups and longitudinal data (e.g., attendance records, persistence indicators) would allow for a more robust evaluation of its impact on early school leaving. Finally, combining qualitative analyses with quantitative measures could provide a more comprehensive understanding of students' learning trajectories.

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5. Problem solving laboratory: thinking in action in disadvantaged contexts

by Pier Luigi Ferrari, Annarita Monaco

The aim of this paper is to present and discuss experiences aimed at promoting problem solving processes in disadvantaged contexts. Some problem solving activities developed in a fifth-grade primary school class are described. The problems are all taken from different editions of the Transalpine Mathematical Rally.

We have focused on some aspects that we believe to be relevant, such as: (1) the criteria adopted for the choice of problems, to try to identify and highlight the characteristics that are able to trigger effective reasoning processes, avoiding strategies based on the search for keywords to identify the required operations; (2) the organization of interaction and communication activities in the classroom, in order to identify the best conditions to promote fruitful collaborative processes; (3) the arguments developed in problem solving and their potential to help students fully understand problem situations and design effective resolution procedures, even neglecting the static models proposed by current literature on the topic.

Scopo di questo contributo è la presentazione e discussione di esperienze finalizzate a promuovere processi di problem solving in contesti svantaggiati. Vengono descritte alcune attività di risoluzione di problemi sviluppate in una classe V di scuola primaria. I problemi sono tutti ricavati da diverse edizioni del Rally Matematico Transalpino. Ci siamo focalizzati su alcuni aspetti a nostro giudizio rilevanti, come ad esempio: (1) i criteri adottati per la scelta dei problemi, per cercare di individuare e mettere in luce le caratteristiche che sono in grado di innescare processi di ragionamento efficaci, evitando strategie basate sulla ricerca di parole chiave per individuare le operazioni richieste; (2) l'organizzazione delle attività di interazione e comunicazione in classe, per individuare le condizioni migliori per promuovere processi

collaborativi fruttuosi; (3) le argomentazioni sviluppate nella risoluzione dei problemi e le loro potenzialità per aiutare gli alunni a comprendere a fondo le situazioni-problema e a progettare procedimenti risolutivi efficaci, anche trascurando i modelli statici proposti dalla letteratura corrente sul tema.

1. Introduction and theoretical background

The main goal of this paper is to present some experiences of development of problem solving skills through interaction and discussion in class or in small groups. The ability to solve problems in everyday situations is regarded as the core of mathematical competence in the Recommendation of the Council of the European Union adopted on 22 May 2018. Classical research on problem solving in Mathematics (e.g, Schoenfeld, 1985) has pointed out that resolution processes involve not only mathematical knowledge and computational skills, but also heuristics (i.e., strategies for making progress in different contexts), control (i.e., global decisions related to the selection and implementation of strategies) and the students' belief systems (about self, the environment, the topic and Mathematics). Therefore, contrary to school and editorial tradition, problem solving should not be aimed at simply applying knowledge and skills to stereotyped situations, but also at developing other competencies, related to heuristic strategies and control processes, bringing students' beliefs to the surface. Consequently, problems should consist in significant questions, related to everyday life, and not just repetitive exercises or questions to which the answer simply requires recalling a definition or a rule or performing a computation. A full understanding of the context of the problem is crucial, since highly context-dependent strategies are typical of good problem solvers, as shown by Lesh (1981). As a consequence, the text of the problem and, more generally, any representation of the problem situation acquire a major role. There is a wide consensus on the importance of representations in problem solving. For example, according to Schoenfeld «the way that one represents a problem, and interprets that representation, is a critical factor in problem solving» (1985, p. 67). The National Curriculum Guidelines reiterate the importance of representing mathematical objects in different ways. Representation can occur through drawings, diagrams or verbal texts. Many researchers (such as Duval, 1993; Polya, 1945) have stressed the value of activities of this kind. The use of different semiotic systems positively influences learning processes on multiple levels. On the one hand, it allows students to separate mathematical concepts from their representations and on the other, it makes the algorithms associated with each of the semiotic systems involved accessible (such as, for example, numerical calculations or graphic methods). Activities of this type are normally carried out in primary school, but, more often than not, the conversion of representations is not given value as a tool for problem solving. This attitude does not seem to be fruitful for the construction of mathematical concepts (Duval, 1993), since it is preferable that the students themselves acquire the mental habit of representing in different semiotic systems and of operating transformations within the same system or from a system to another. Among the semiotic systems, verbal language plays a central role because of its enormous potential, as argued by Hasan (2011). So the interpretation of the text of the problem and the related activities become central, even if, unfortunately, they are often not regarded as a mathematical activity. Actually, the work on the text of the problem (interpretation, reformulation, comparisons) involves all the components identified by Schoenfeld: resources, heuristics, control and beliefs.

More detailed discussions on the relationship between problems and pupils' experiences and on the difficulties arising in the interpretation of the text of problems have been developed by Zan (2007, 2016a, 2016b) and Ferrari (2021).

Our experiences involve argumentations too. For some years now, a keen interest in argumentation in Mathematics has developed among researchers in Mathematics education. Although this interest has clear roots in problem-solving activities carried out in the past (for example, in requests to motivate the resolution procedures adopted), the theoretical reflections that followed have focused above all on the relationship between argumentation and proof and on the theme of continuity or discontinuity between the two processes (see for example Mariotti, 2022).

In this perspective, the emphasis is mainly on the validation of knowledge already acquired, while, in a school context of problem solving, argumentation cannot fail to play the function of generating new knowledge. This limit, shared by almost all current models of argumentation[1], has been highlighted and discussed in depth by Hintikka (1999) who proposes a logic aimed at investigation. Arzarello and Soldano (2019) have discussed examples of use of the logic of investigation in the classroom, albeit in a context of initiation to proof. We are interested in a perspective on argumentation that includes its use in problem solving, not only to motivate the solutions achieved but also as a contribution to achieving them. Another shortcoming of current models of argumentation, including Toulmin's, is that they are mainly normative and

[1] We refer, for example, to the Toulmin' s model (Toulmin, 2003). See Ferrari (2024) for some criticism.

product-oriented, neglecting the interactive processes that develop among students. From an educational perspective, such processes are more relevant than the product itself, and often proceed in different and unpredictable ways. According to Catarina Dutilh Novaes (2020) argumentation (and proof) activities are essentially dialogic. Similar ideas have been proposed by Mercier and Sperber (2019) and Ferrari and Saccoletto (2024). So, the development and improvement of arguments is promoted by interactions among peers rather than comparisons with pre-established models. The uselessness of relying on pre- established models is also motivated by the fact that an argument formulated in a language is never completely explicit, since it is not possible to define all the words used, and the level of explicitness to be required must be linked to the educational objectives. From the point of view of learning, what matters is not producing arguments that correspond to a pre-established model, but being able to improve an argument when some critical issues are highlighted.

2. Which problems?

We must therefore move beyond the idea of a problem as a container of operations, in which the meaning of the situation described matters little, and even the text describing it is not intended to help understand the situation but only to identify the operations required, possibly through the insertion of keywords. From all this it follows that the different ways of interpreting problems require radically different and contrasting performances from students.

In the activities we present, understanding and representing the problem situation are fundamental.

Furthermore, we ask students to bring into play not only their mathematical knowledge and skills, but also their personal experience and their knowledge of the world, which are obviously related to heuristics, control and beliefs systems. For this reason, we always try to use problems inserted in situations familiar to students, in order to allow them to connect them to their experiences and to bring into play their knowledge. This, of course, does not exclude playful or fantastic situations. The language used must also avoid unnecessary complications and all implicit terms that can generate or consolidate stereotypes. For example, in the problem "Johnny puts his 56 toys in 4 boxes. How many toys does he put in each box?" it is left implicit that the number of toys placed in each box is the same, because otherwise the problem would not have a single solution. The text of the problem therefore reflects the stereotype, which is sometimes found in school books, according to which "divide" is equivalent to "divide into equal parts". The text in quotation marks evidently does not allow the student to reconstruct the situation (which is in any case highly unrealistic), but it is sufficient to suggest that the operation required is a division.

The interpretation of the text in this way becomes an integral part of the resolution process, which requires time and can be enriched by moments of discussion and comparison between students. From the point of view of the mathematical structure, it is appropriate to include problems that require some modeling and are not solvable by simply applying one or more operations to numerical data. This includes also problems allowing no solution or more than one. These problems are more suitable for triggering discussions and arguments and for reflecting on the meaning of the operations. The availability of an algorithm allowing students to find the solution immediately in most cases inhibits further discussions and reflections on the meanings involved.

3. The sample and the learning environment

The research sample consists of a class of 20 students, 8 females and 12 males of a fifth-grade primary school, in a school located on the outskirts of Rome. One male and one female student are followed by the support teacher; the female student is also supported by a communication assistant and an OEPAC educator. Starting from the first grade, attempts were made to create a challenging and engaging learning environment in the classroom: some special courses were designed and then implemented: on combinatorial calculus, on relational thinking, on probability, on problem solving, both in person and remotely, with the intervention of university professors and some graduates. In particular, the problem-solving work, scheduled on a weekly basis, engaged the students, divided into groups, in solving challenging problems, not just exercises: they analyzed the texts, and then researched, communicated, negotiated and applied original strategies and representations. The students constantly self-assessed themselves, with respect to the management of collaborative work, but also to the difficulties encountered and their perceived evolution as problem solvers. They also simulated competitions, lasting 50 minutes, based on the rules that the Transalpine Mathematical Rally Association has established for its competitive tournaments. The rules also require that the groups of students are responsible for solving a certain number of problems (six for the fourth grade and seven for the fifth grade), and that the groups flexibly collaborate with each other, as the management of the task is the responsibility of the whole class. Well beyond the competition, an engaging environment of active research for the solution of the problems faced was created in the class, with attention to the processes, as well as the products, which continued, at times, even beyond the scheduled time. Producing inaccurate or poorly developed arguments has never been considered a failure: it is precisely from these limitations that we could start again to relaunch discussions aimed at giving rise to new confrontations and resolution procedures.

The dialogic nature of arguments suggests allowing students to experiment with situations in which different ideas are compared, with the aim of choosing those deemed most suitable and of refining the analyses of the situations. Students, especially the younger ones, once they have found a solution that more or less corresponds to the requests, do not spontaneously feel the need to consider potential objections or to show that their solution is the only possible one. This can limit the depth of students' analysis of the problem situation. If a ground for dialogue does not arise spontaneously (for example because students immediately converge on an acceptable solution), it may be educationally appropriate to propose an artificial one to stimulate students' thinking. For this reason, in cases where the discussion has not developed sufficiently, we decided to relaunch the problem in the following days by proposing to students to discuss an answer, generally incorrect, attributed to fictional students from other classes.

4. Some examples

4.1. Large and small glasses [(Rally Mat. Tran. 21, 3-5) ©ARMT]

Giulia is organizing a birthday party for her little brother. She buys several bottles of orange soda. The contents of one bottle can fill 5 large glasses or 8 small glasses. During the party Giulia serves 23 large glasses and 26 small glasses of orange soda, opening as few bottles as possible.

How many bottles did Giulia have to open? Explain how you found your answer.

To solve this problem, students should understand, during the first attempts, that neither the order in which the glasses are filled nor which bottle they are filled with are important. Instead, they should focus on the fact that Julia has filled 23 large glasses and 26 small glasses, and that 5 large glasses correspond to one bottle, as do 8 small glasses.

This problem could be solved by calculating the expression $23 \times 1/5 + 26 \times 1/8$ which gives a result of 7.85 and allows us to answer the question. However, this use of fractions is premature at the primary school level, also considering that the students in the sample live in disadvantaged contexts. There could be different resolution models: one could represent the 23 large glasses, form groups of 5 with them, each equivalent to a bottle, then count these groupings and find that they correspond to 4 bottles with the remainder of 3 large glasses. Similarly, one could group the 26 small glasses into groups of 8 and find that another 3 bottles are obtained with the remainder of 2 small glasses. And this is the resolution model chosen by a group of students in our sample. Alternatively, one could use multiples of 5 and 8 to get closer to the number of glasses served, and then understand that with 4 bottles you can fill 20 large glasses (4×5) and with the other 3 bottles 24 small glasses (3×8); finally we can deduce that with 7 bottles there are 3 large glasses and 2 small ones left to fill with an eighth bottle.

We present the solution of a group of fifth-grade students involved in the research. This, like other examples, is presented with the aim of highlighting some crucial aspects of our approach.

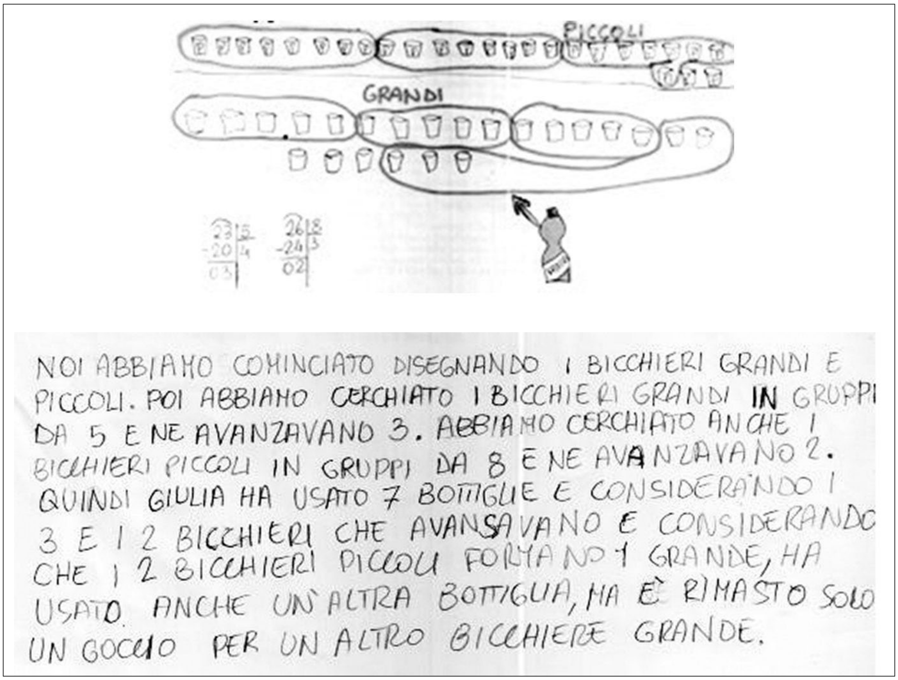


Fig. 1 – An example of answer

We started drawing the large and small glasses. Then we circled the large ones into groups of 5, and there were 3 left. We circled the small ones too into groups of 8 and there were 2 left. So Giulia used 7 bottles and considering the 3 and 2 glasses left and considering that two small make one big, she used 1 bottle more, but it was left just a drop for another big glass.

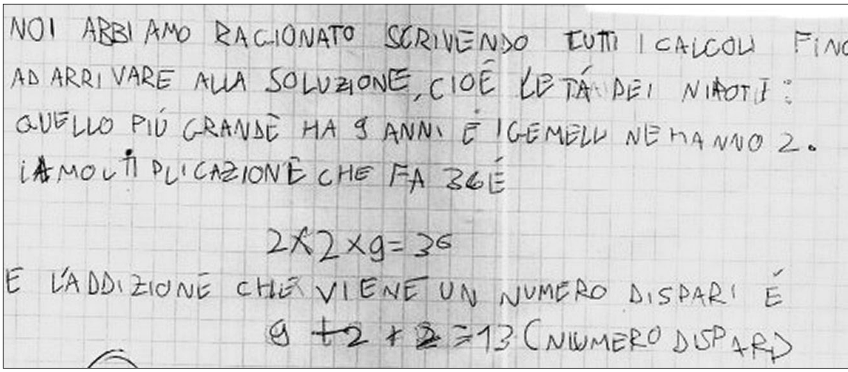
It is interesting to note that the students felt the need to evaluate the amount of orange juice left. This is not required by the problem but is an implicit question in the narration. Regarding the amount left, the students reasoned approximately, evaluating it as equal to a large glass. This answer is completely acceptable in a context of Mathematics applied to reality. Requiring the exact answer (that is 3/20 of a bottle, i.e., a little less than a large glass and a little more than a small one) would detach the problem from reality: the relevant information is that to put the leftover orange juice you need a large glass, and in reality no one would dream of calculating exactly the fraction of orange juice left.

4.2. Uncle and nephews [30° RMT May 2023 – 9 ©ARMT]

Bruno has three grandchildren, two of whom are twins. Adele asks him how old each child is. Bruno replies: "If you multiply their ages you get 36 and if you add them you get an odd number; the eldest has red hair". Tell us how old the three children are. Explain your reasoning and give the details of the calculations.

This problem is not a genuine real-life problem but illustrates a hypothetical riddle proposed by Bruno. The situation should be familiar to the students as it involves kinship ties (uncle, nephews, twins) and whole numbers. The solution requires careful reading of the text and the coordination of different information (product of the ages, the fact that the sum must be odd, existence of twins who are the youngest). The students should understand that two of the three numbers must be equal and therefore the possible triplets are: 1-1-36; 2-2-9; 3-3-4; 1-6-6; they should also understand that, of the four previous triplets, only two must be considered, as the sum of the ages, as specified in the text, must be odd: 2-2-9 and 1-6-6. In the third and final analysis, they should understand from the sentence "the eldest nephew has red hair" that the non-twin nephew is the oldest and therefore the correct triplet is: 2-2-9. Alternatively, one could proceed by trial and error, looking for three numbers, two of which are equal and which satisfy the conditions set.

Let's now see some excerpts from the response of a group of students.


NOI ABBIAMO RAGIONATO SCRIVENDO TUTTI I CALCOLI FINO AD ARRIVARE ALLA SOLUZIONE, CIOÉ L'ETÁ DEI NIPOTI: QUELLO PIÚ GRANDE HA 9 ANNI E I GEMELLI NE HANNO 2. LA MOLTIPLICAZIONE CHE FA 36 É

2×2×9=36

E L'ADDIZIONE CHE VIENE UN NUMERO DISPARI É

9+2+2=13 (NUMERO DISPARI)

Fig. 2 – An example of answer

We reasoned by writing all the computations, until we got to the solution, which is the age of the nephews. The eldest is 9 years old, and the twins are 2 years old. The multiplication giving 36 is 2 × 2 × 9 =36 and the addition giving an odd number is 9 + 2 + 2 = 13 (odd number).

In the motivation given, the students verify that the solution they are proposing corresponds to the requests, but they do not highlight that it is the only possible one. In particular, they do not explain why the twins are the two youngest. It is possible that they understood it, but they did not report it in their explanation. To push the students to refine the argument, the problem was relaunched a few days later, in the following terms: the teacher stated that a student from another class found that the ages of the children are 6, 6 and 1. Proposing an alternative (wrong) solution had the function of creating a dialogic context that would allow the students to deepen their analysis.

Here are some excerpts from the discussion (recorded in audio):

The problem says that the eldest has red hair, the eldest is one, not two. Otherwise they would have had to write that the two eldest have red hair, that is, the twins.

The eldest is not anyone's twin. That student is wrong because in his solution there are two older brothers, not one.

He didn't read well that the eldest has red hair.

The students immediately respond by focusing on the expression "the eldest". The promptness with which they indicated it suggests that they had used the information implicitly contained without making it explicit.

4.3. The strange pizza [(Cat. 5, 6) ©ARMT.2004 – 12°]

To break a record, the inhabitants of a village decide to make a large rectangular pizza. The pizza must be 4 m long and be composed of four parts: one with mushrooms, one with ham, one with olives and one with vegetables. To accommodate everyone's tastes, the inhabitants decide that:

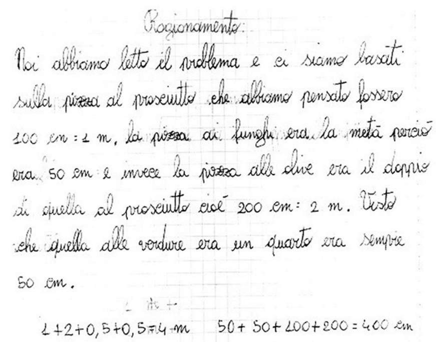
– the length of the part with ham must be double that of the part with mushrooms and half that of the part with olives;
– the length of the part with vegetables must be a quarter of the longest part.

What will be the length of each part of the pizza?

Explain how you found the solution.

The students must understand that the part with olives is the longest, as well as understand the different ratios between the lengths and compare them with the longest part: the length of the pizza with ham is half the length of the pizza with olives; the length of the pizza with mushrooms is equal to the length of the pizza with vegetables and a quarter of the pizza with olives. They should then deduce that the length of the pizza with the olives is equal to half of the entire length (4 m), therefore 2 m; consequently the other lengths are: 1 m, 50 cm, 50 cm. Alternatively, they could represent the ratios graphically and arrive at the answer.

Here is an excerpt of the solution model created by one of the groups.


Ragionamento:
Noi abbiamo letto il problema e ci siamo basati sulla pizza al prosciutto che abbiamo pensato fosse 100 cm = 1 m, la pizza ai funghi era la metà perciò era 50 cm e invece la pizza alle olive era il doppio di quella al prosciutto cioè 200 cm = 2 m. Visto che quella alle verdure era un quarto era sempre 50 cm.
1+2+0,5+0,5=4 m   50+50+100+200=400 cm

Reasoning
We read the problem and based ourselves on the pizza with ham, and we thought it could be 100 cm = 1 m, the pizza with mushrooms was the half so it was 50 cm, and instead the pizza with olives was double the one with ham, which is 200 cm = 2 m. As the one with vegetables was one quarter, it was still 50 cm.
1 +2 + 0,5 + 0,5 = 4 m
50 + 50 + 100 + 200 = 400 cm

Here too, it seems that the students found the solution through numerical trials. The explanation describes the successful trial (perhaps the only one carried out) with numerical verification but without further explanations. In this case too, the problem was relaunched in the following days. The teacher states that the students of another class found that the part with the mushrooms is 30 cm long. Here are some excerpts of the discussion (recorded in audio):

If the part with the mushrooms is shorter, then the part with the olives should be longer, it's a question of logic.

If the part with the mushrooms is only 30 cm long, then 20 cm are left with nothing.

Maybe they wanted to do 30-70 instead of 50-50.

We have to stick to what the problem says... the part with the mushrooms must be a quarter of the part with the olives.

In this phase, the students mostly compared the answer reported by the teacher with their own. Some pointed out the need to lengthen one of the other parts or the presence of 20 cm without anything on top. Only few, as in the last intervention reported, highlighted that the configurations with 30 cm of mushrooms did not respect the conditions of the problem.

5. The hidden numbers [(Cat. 3, 4, 5) ©ARMT]

Two butterflies landed on Arianna's open notebook and hid two numbers.

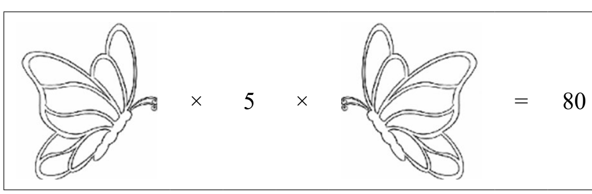


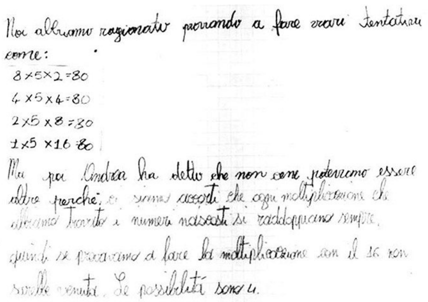
Fig. 3 – The hidden numbers

Now in your notebook you can see only the numbers 5 and 80, two "×" signs and a "=" sign. The hidden numbers are both whole numbers and could be the same or different from each other. What could the two hidden numbers be? List all the possibilities and show how you found them.

In this problem one is asked to examine the given operation, understand that it is made up of four numbers, two of which are hidden, and that it is an equality between 80 (on the right) and the result of two successive multiplications (on the left) in which the first factor is missing, the second is known, and the third is missing. You must understand that it is a matter of choosing a first number, multiplying it by 5, calculating the product, and multiplying the latter by a second number to be found, to obtain 80.

Students can proceed by trial and error, for example by inserting 1 in the first space and obtaining 5 as the first product (1×5); or by inserting 2 as the third factor, obtaining 1×5×2=10; or by performing the inverse operation of the multiplication (80: 5) and obtaining 16 which will be the third factor; but other attempts could be made. They could instead understand that, by dividing 80 by 5, the product of the other two numbers sought will be found: 16; therefore, all the divisors of 16 are sought to correctly form the pairs of factors to obtain it: (1; 16); (2; 8); (4; 4), (8; 2), (16; 1).

Below is an example of a response given by a group:


Noi abbiamo ragionato provando a fare vari tentativi come:
8×5×2=80
4×5×4=80
2×5×8=80
1×5×16 80
Ma poi Andrea ha detto che non ce ne potevano essere altre perché ci siamo accorti che ogni moltiplicazione che abbiamo trovato i numeri nascosti si raddoppiano sempre quindi se proviamo a fare la moltiplicazione con il 16 non sarebbe venuta. Le possibilità sono 4.

We made various trials such as
8 × 5 × 2 = 80
4 × 5 × 4 = 80
2 × 5 × 8 = 80
1 × 5 × 16 = 80
but then Andrea said that there were no more, because we noticed that in any multiplication we found the hidden numbers always double, then if we try to multiply by 16 it will not work. The possibilities are 4

When the problem is repeated, the students immediately realize that there are 5 possibilities and not 4. The teacher then tells them that a student from another class discovered that under the first butterfly there is the number 3. Here are some excerpts from the discussion (recorded in audio):

It can't be!
Why not?
Because you can't get to 80. I mean, 80 is not in the three times table.
If from 3 you could get to 10, or even 20, then it would be possible, but neither 10 nor 20 are in the three times table.

Here divisibility by three is identified with the occurrence in the three times table. Being asked to consider a different solution (correct or incorrect) allows them to see mathematical relationships they had not previously considered, such as the fact that the divisibility of 80 by 3 would be guaranteed by the divisibility of 10 or 20 by 3.

6. Conclusions

From the observation of the students, in occasion of the group interactions, it seemed evident how much they tried to tune in with the thoughts of their classmates, overcoming a certain tendency of the solvers to mainly follow the thread of their own reasoning; or, at least, discussing at the end of the work, self-analyzing, they became aware of their solving individualisms. An attitude of research for communication was perceived: they listened, observed, looked for the most appropriate words, signs, or gestures to make themselves understood by others, but also to specify, punctuate, and deepen their solving elaborations. The challenging situations were managed with ever greater determination, easing the fear of not knowing how to proceed, stimulated, indeed, by a healthy tension to think and produce. The mathematical knowledge, built or under construction, in Mathematics hours, was applied creatively in new situations; its value was explored, testing its limits. In some problems, such as "Large and small glasses", the role of multiple representations clearly emerged, for the understanding of the problem and the start of a solution strategy, but also for connecting the problem to the previous experiences of students and allowing each of them to communicate freely. In this regard, also in the other problems discussed the importance of choosing problem situations that are familiar to the students is confirmed, in order to allow them to bring their contextual knowledge into play. The management of the problems in class also showed how even students' initially incomplete, imprecise or incorrect arguments were useful for setting in motion thought processes to subsequently reach more complete and correct forms.

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6. Inclusive education, learning loss and implicit dropout: how to interpret a relationship

by Elisabetta Robotti, Alessandra Boscolo

Implicit school dropout is a significant educational challenge, influenced by social inequalities and learning disengagement, particularly in Mathematics. This study explores the relationship between implicit dropout, learning loss, and inclusive education, highlighting the role of inclusive education in preventing student disengagement from the early years of schooling. By integrating the Universal Design for Learning (UDL) framework, the research highlights two key principles for effective instructional design: accessibility through multimodal approaches and meaningfulness via problem-centered education. A learning sequence on fraction in primary school illustrates how these strategies can create equitable learning opportunities, particularly for students at risk of learning difficulties. It represents a mathematical experience to counteract learning loss and school dropout in disadvantaged contexts. Indeed, this approach may be of central relevance for students with disadvantaged background.

Il fenomeno dell'abbandono scolastico implicito è una sfida educativa significativa, influenzata dalle disuguaglianze sociali e dal disimpegno nell'apprendimento, in particolare nella Matematica. Questo studio esplora la relazione tra abbandono implicito, basso apprendimento e educazione inclusiva, evidenziando il ruolo dell'educazione inclusiva nel prevenire l'abbandono implicito degli studenti fin dai primi anni di scuola. Sfruttando il quadro dell'Universal Design for Learning (UDL), la ricerca sottolinea due principi chiave per un design didattico efficace: l'accessibilità attraverso approcci multimodali e la rilevanza tramite un'educazione centrata sui problemi. Una sequenza di apprendimento sulle frazioni nella scuola primaria illustra come queste strategie possano creare opportunità di apprendimento equo, in particolare per gli studenti a rischio di difficoltà di apprendimen-

to. Essa rappresenta un'esperienza Matematica per contrastare l'apprendimento poco significativo e l'abbandono scolastico nei contesti svantaggiati. Questo approccio potrebbe essere rilevante anche per gli studenti con background svantaggiato.

1. Introduction

The issue of school dropout – both implicit and explicit – is often addressed reactively, intervening when students show signs of disengagement and struggle that put their educational journey at risk. However, as highlighted by research in education, it is possible – and necessary – to act preventively, not by merely addressing individual cases of fragile learners but by designing educational contexts that inherently reduce the risk of dropout. This requires a shift in perspective: rather than focusing solely on remediation, we must work to ensure accessibility and meaningful learning experiences for all students, including those from disadvantaged backgrounds. Indeed, according to Benvenuto (2016), implicit dropout is an indicator of educational inequality and lack of equity, as it disproportionately affects students from socioeconomically disadvantaged backgrounds.

Therefore, one of the most effective ways to tackle this challenge is through the implementation of inclusive teaching approaches, starting at the primary and even the pre-school level, that can engage, support, and guide the learning process for a diverse range of students. The international educational framework also supports this inclusive perspective. The UN Agenda 2030 defines 17 Sustainable Development Goals (SDGs), among which Goal 4 focuses on the right to education, calling on nations to ensure quality, inclusive, and equitable learning opportunities for all students.

Research also suggests that implicit dropout is linked to learning loss, particularly in Mathematics, as highlighted in national assessments such as INVALSI reports (INVALSI, 2022). In particular, looking at Mathematics education from the early years of schooling is particularly relevant, as far as research findings suggest the critical role of primary school Mathematics education in shaping long-term learning outcomes. Then, as researchers in Mathematics Education, we contribute to this global effort by focusing on teacher training and instructional design, ensuring that Mathematics teaching aligns with the principles of equity, quality, and inclusivity outlined in the UN Agenda, starting as early as possible.

At the core of an inclusive approach are two essential principles: adapting for including, i.e. teaching must be flexible and responsive to student's diverse needs, and considering diversities as opportunities, i.e. diversity should be embraced as a resource for learning. From our perspective, the two key directions for promoting inclusive instructional design include ensuring that all students can actively engage with mathematical concepts (Accessibility) and that learning is relevant and motivating for each student (Significance). To effectively follow these two directions in Mathematics education is necessary the integration of multimodal approaches, according to the embodied cognition perspective (Núñez and Lakoff, 2005), and problem-centered education, following the principle of necessity (Harel, 2008): students should experience the need to learn to be engaged in this effort.

To implement these principles, we rely on the Universal Design for Learning (UDL) framework (CAST, 2011) as a guiding model for inclusive instructional design. Our work takes place within a research-action group, the "Mathematics Teaching Laboratory" at the University of Genoa[1], where we explore collaborative design methodologies (Morselli and Robotti, 2023). This approach integrates findings from Mathematics Education, neuroscience, and cognitive sciences to develop inclusive teaching practices.

To illustrate this approach, we will present a learning sequence on fractions, designed for primary school students, which reflects our vision of inclusive Mathematics education through the lens of Universal Design for Learning (UDL). By leveraging multimodal activities and problem-based tasks, this pathway fosters accessible and meaningful learning experiences, ensuring that all students – regardless of their background or learning profile – can build strong and lasting mathematical understanding. The example gives shape to our inclusive, ensuring that Mathematics education is accessible and meaningful for every student from the very beginning of their educational journey, as required by the learning objectives set in the *Indicazioni nazionali*.

We emphasize that, although the paper presents in detail on a learning sequence on fractions, the proposed approach is not content-specific and can be extended to other key mathematical concepts which share similar cognitive and didactical challenges (e.g., Robotti and Boscolo, 2026).

From a methodological perspective, the learning sequence was implemented in a primary school context within the presence of students coming from a low socio-cultural background or first-generation immigrant families, as part of a broader design-based research approach. The intervention was documented through classroom observations, students' written productions and notes from teachers' reflections, which were qualitatively analysed to

[1] https://sites.google.com/view/labddm/home?authuser=0.

explore how accessibility and meaningfulness supported students' engagement and learning.

2. Implicit dropout, fragile learning, and Mathematics education

Implicit dropout refers to a phenomenon in which students, although formally enrolled in school, experience a progressive disengagement from learning. This disengagement manifests in low academic performance, lack of participation, and minimal educational progress, ultimately leading to a failure to acquire fundamental competencies. Unlike explicit dropout – where students physically leave school – implicit dropout occurs within the school system, making it more difficult to detect and address (INVALSI, 2022).

Although the overall implicit dropout rate at the end of high school shows a slight decrease, it is increasing significantly among students from disadvantaged backgrounds, e.g. in southern Italy. Among these students, the risk of dropping out – whether implicit or explicit – has at least doubled and is already evident by the end of lower secondary school.

The COVID-19 pandemic has shed light on the complex relationship between learning loss and implicit school dropout, particularly in disadvantaged socio-cultural contexts. The sudden shift to remote learning, coupled with disparities in access to educational resources, has exacerbated pre-existing inequalities, leaving many students without meaningful learning opportunities (UNESCO, 2021). This situation has raised critical questions about the conditions necessary to ensure equitable and effective learning experiences: To what extent is learning loss related to learning opportunities? What forms of teaching and learning can counteract this phenomenon?

According to the INVALSI National Report (2022, p. 54), fragile learning – both in literacy and numeracy – is closely linked to implicit school dropout.

While dropout rates are typically analyzed in the context of secondary education, particularly upper secondary education, data suggest that implicit dropout has its roots much earlier in the educational pathway. Moreover, research highlights that the persistence of fragile learning over time is a key factor contributing to dropout. This underscores the urgency of early intervention through appropriate teaching strategies, which specifically address learning vulnerabilities. As Ricci (2019, p. 8) notes, the early signs of school dropout begin to manifest as early as primary school, even though they often escape official statistics due to the difficulty of identifying and quantifying them. He emphasizes also that: «Reaching inadequate levels of preparation is one of the most significant causes of school dropout, making it essential to monitor this phenomenon from its earliest manifestations» (translation by the authors[2]). Therefore, any strategy aimed at reducing school dropout must not focus solely on secondary education but should instead prioritize early educational interventions (ivi, p. 9). Indeed, as the researcher suggested, «prevention of school dropout precisely at the moments when intervention is most likely to be successful» (ivi, p. 8; translation by the authors[3]).

The strong link between implicit dropout and students' perception of their effectiveness in Mathematics is particularly concerning. When students experience difficulties in Mathematics and achieve low performance, their sense of self-efficacy declines, increasing the risk of disengagement. This highlights the necessity of identifying and addressing mathematical difficulties at their earliest signs, if not before they even emerge. Indeed, the INVALSI National Report (2022) explicitly states: «It is crucial to pay specific attention to primary school outcomes, as even small differences at this stage can lead to significant difficulties in later grades» (translation by the authors).

INVALSI data from 2023 further support this concern: results in Mathematics for second-grade primary students are significantly lower than those recorded in 2019 and 2021 and remain in line with the declining trend observed in 2022. Similarly, fifth-grade students have performed worse than in previous years, including 2022, 2021 and 2019. These findings indicate that mathematical difficulties do not simply arise in later schooling but can already be detected in the early years of primary school. These findings from the INVALSI Mathematics tests in the early years of primary school may suggest that the first symptoms could even be identified in preschool. Research, such as that of Noël (2009) on finger counting, points out that early mathematical struggles may have roots in preschool experiences.

In conclusion, preventing and combating school dropout has become a priority, particularly for students from disadvantaged backgrounds and especially in Mathematics, a subject often seen as a gateway to future academic and professional opportunities. However, it is not sufficient to merely monitor students' learning; it is necessary to design inclusive learning environments. Then, the challenge focuses on the issue: *How can we design effective teaching strategies to address fragile learning?* A potential answer lies in the principles outlined in Goal 4 of the United Nations 2030 Agenda, which

[2] Original quotation: «Il raggiungimento di livelli di preparazione inadeguati rappresenta una delle cause più importanti della dispersione scolastica e quindi è fondamentale monitorare il fenomeno sin dalle sue prime manifestazioni».

[3] Original quotation: «prevenzione alla dispersione scolastica proprio nei momenti in cui l'intervento può avere maggiore probabilità di successo».

calls for «Quality, inclusive, and equitable education at all levels and for all». Indeed, research on inclusive education has shown positive effects on academic achievement for all students (e.g., Demeris, Childs and Jordan, 2008).

By fostering an inclusive educational approach, we can create learning environments that are accessible, meaningful, and effective for every student, from the earliest years of schooling. The goal is not only to support students at risk of implicit dropout but also to ensure that all learners develop strong, transferable mathematical foundations that empower them to succeed and, therefore, prevent implicit dropout. But *how to do it?* Our aim is to propose an approach and describe how this model could serve as a design framework for learning sequence potentially inclusive, and then, according to our statements, potentially preventing school implicit dropout.

In the following paragraphs, after clarifying our inclusive perspective, we propose two guiding principles and show how Universal Design for Learning (UDL) can support teachers in designing and implementing inclusive Mathematics teaching and learning. Our aim is to outline an approach and describe how UDL may serve as a design framework for potentially inclusive learning sequences. Available evidence suggests that UDL-informed interventions are associated with improved learning outcomes, as indicated by the meta-analysis carried out by King-Sears and colleagues (2023) and, in Mathematics-focused studies, with higher achievement and/or engagement and the reduction of instructional barriers that can limit access and participation (Root *et al.*, 2020; Yavuzarslan and Arslan, 2020). Taken together, these findings support the hypothesis that UDL may contribute to conditions that foster participation and persistence in Mathematics – for instance by strengthening engagement and access – thereby plausibly mitigating risks of disengagement and withdrawal that can precede forms of implicit school dropout, while acknowledging that direct evidence on dropout outcomes remains limited.

3. Inclusive education

To address the important questions of how to develop inclusive education and how to design appropriate teaching that responds to the difficulties in Mathematics learning, we turn to the goals of the UN 2030 Agenda, specifically Goal 4. This goal, focused on the right to education, urges nations to commit to ensuring quality, inclusive, and equitable education at all levels and for all. This global perspective offers an essential compass to guide our inquiry, emphasizing the importance of a systemic and universal approach to overcoming barriers to learning and fostering a more just and inclusive society.

Therefore, an effective approach to achieving this aim is grounded in the implementation of inclusive teaching approaches in schools. These approaches, implemented as early as possible, should be capable of accommodating, supporting, and guiding the learning process of a diverse range of students. This goal, aimed at providing equal learning opportunities, also addresses school dropout (both implicit and explicit), a key indicator of educational inequality (Benvenuto, 2016).

We will therefore try to investigate what "inclusive, equitable and quality education" means.

In the Italian context, the notion of inclusion has progressively evolved from a focus on placement and special needs towards a broader perspective that addresses the educational and social needs of all students within the classroom (Nilholm *et al.*, 2017).

For this innovative perspective to be effective, a supportive school community is essential to sustain the educational process (teacher training, school support, collaboration with families, etc.). This represents the current *"broad"* perspective that characterizes the Italian school system in relation to the concept of inclusion.

Adopting the perspective that "inclusion" encompasses meeting the social, educational, and instructional needs of all students, we align with the definition proposed by the UNESCO International Bureau of Education (2009): «An ongoing process aimed at offering quality education for all while respecting diversity and the different needs, abilities, characteristics, and learning expectations of students and communities, eliminating all forms of discrimination» (p. 18).

Moreover, the UNESCO (2017) document Guide for Ensuring Inclusion and Equity in Education further defines inclusion as «a process that helps overcome barriers limiting the presence, participation, and achievement of learners» while emphasizing that equity «is about ensuring that there is a concern with fairness, such that the education of all learners is seen as having equal importance».

The interest in school inclusion has also been explored in educational research. To examine this aspect, we refer to the literature review conducted by Loreman and colleagues (2014), which focused on European studies analyzing the implementation of inclusive education in school practices.

The approach adopted by most European studies examines inclusive education across three levels of intervention – micro, meso, and macro – encompassing respectively individuals and classrooms, schools and their operating contexts, and broader systems such as local and national government bodies.

Among the objectives of some of these studies was to define indicators to measure the effectiveness of an inclusive system. Specifically, the indicators related to the effective implementation of school inclusion in teaching processes at the micro level (which is the most relevant in this context) are: climate, school practices, classroom practices, collaboration and shared responsibility, and support for individuals. So, the question is: *Do we have tools that allow us to design inclusive teaching?* In other words, tools that enable us to assess whether teaching is implemented rigorously, and in alignment with the values, objectives, and definitions outlined above. Tools that help measure whether, and to what extent, it is effective in achieving learning objectives, engaging all students, and providing support for each of them.

To promote inclusive education, international documents such as UNESCO-IBE's *Reaching out to all learners* (2021) identify key actions at the school level, including understanding contextual challenges and resources, strengthening inclusive leadership, and, above all, investing in teacher training. In line with this perspective, we focus on teachers' professional development and the adoption of inclusive educational approaches, particularly continuous training and the use of Universal Design for Learning (UDL) as a framework for designing high-quality, inclusive teaching. We refer to Section 5 for the methodological aspects related to the use of UDL.

Additionally, the document identifies eight indicators that can help teachers assess the inclusiveness of their teaching approach.

These indicators emphasise planning with all students in mind, active participation, mutual support, and assessment practices that promote learning for every student. We can observe how they align with those identified in the literature review conducted by Loreman and colleagues.

In accordance with the UDL framework and these indicators, it is evident that promoting inclusive education requires a new perspective: teachers must view diversity as an opportunity, leveraging it to design inclusive education. In other words, they should adapt the context to respect differences rather than compensate for a deficit to bridge a gap. To understand the position, we take within this perspective, we refer to some key figures in Italian education and pedagogy to show how the idea behind the approach we present is not as innovative as it might seem. In particular, we refer to figures such as Maria Montessori, Emma Castelnuovo, and Franco Lorenzoni, whose teaching methods share common elements: the use of visual representations, hands-on tools, body movement, and physical interaction as means for constructing mathematical knowledge. As we will see in the following paragraph, this perspective is closely aligned with the concept of multimodality. Furthermore, they propose situations that are meaningful to students – situations that are both significant and authentic for them. Their approach aims to provide access to meaningful learning experiences for all by offering a problem-centered education. This leads to two key principles: designing accessible teaching and learning by adopting multimodal approaches and creating authentic problem situations that allow students to engage meaningfully with mathematical concepts. We emphasize that the principles mentioned are not tied to any specific mathematical content.

4. The two-track for inclusion in Mathematics education

Inclusion is accessibility to a meaningful experience of learning for everyone. Our educational vision can be addressed following key guiding principles. The first is to design an accessible teaching-learning environment, looking at multimodal approaches, and the second is to design authentic problem situations to have a significant experience of Mathematics. To do this, we need two essential ingredients: multimodality and meaningful problem situations.

On the one side, the relevance of perception and body movement in the exploration and construction of mathematical meanings is considered more and more relevant in Mathematics education. The roots of this tradition can be traced back to the early 1900s. However, the centrality of perceptuo-motor involvement in Mathematics teaching and learning process sprung out in past decades, when research on Embodied Cognition demonstrated how mathematical thinking is closely connected to bodily experiences, sensory perceptions, and motor actions (Núñez and Lakoff, 2005). Studies in Mathematics Education show that mathematical meaning is constructed through multimodal activities: doing, touching, moving, hearing, and seeing are essential from the early stages of learning to advanced mathematical reasoning (Arzarello *et al.*, 2007; Nemirovsky, 2003; Radford, 2006). Therefore, activities in which students are engaged with their bodies and movements to experience mathematical meaning assume an increasing relevance and the challenge of its implementation in school has been highlighted (Boscolo, 2024).

On the other side, we refer to meaningful and authentic problem situations as situations designed according to Harel's Necessity Principle: «For students to learn what we intend to teach them, they must have a need for it, where "need" is meant intellectual need, not social or economic need»

(Harel, 2007, p. 274). *What do we mean by intellectual need?* In the example we will discuss to present this idea concerns a problem-centered situation in a kindergarten where learning the new concept of "measure" is perceived as necessary for the children. This example does not pertain to the learning path on fractions (as anticipated at the beginning of the article) which we will examine in the following sections and it is not intended as an empirical case study, but as a conceptual illustration of the necessity principle guiding our instructional design.

The teacher needs to create a meaningful situation for the children in which they can construct the meaning of measurement and that allows them to answer the question "Why measure?" as an intellectual need of the children. The problem situation arises from the "Capture the Flag" game.

The teacher suggests playing "Capture the Flag" in an unfair situation where one of the two teams is disadvantaged. The children of that team point out this disadvantage and request a fair situation.

The teacher asks the children what they mean by a "fair situation". Children answer by saying the child with the red flag (Fig. 1) should stay "in the middle". Therefore, the issue shifts to defining "middle": middle means "half", say the children, it also means "dividing into two parts". The children state that, for it to be considered "in the middle", the two parts must be equal, thus characterizing the concept of "half" by adding to the condition "divided into two parts" the requirement that "the two parts must be equal".

Fig. 1 – Capturing flag game in a pre-school: unfair situation (left), fair situation (right)

The proposed activity is designed to highlight the *need for measurement*. Students directly experience the *sense of measurement*, understanding why measuring is important. Afterwards, the teacher can introduce *how to construct two equal lengths*, connecting the practical experience to mathematical concepts related to measurement.

5. Instrument for teaching design: UDL

As mentioned earlier, the UNESCO document *Reaching out to all learners: A resource pack for supporting inclusive education* (UNESCO-IBE, 2021) suggests instructional design based on the Universal Design for Learning (UDL) model to make learning inclusive. CAST (Center for Applied Special Technology) is an educational research and development organization that played a fundamental role in the development and promotion of Universal Design for Learning (UDL) framework to ensure that all educational experiences are intentionally designed to enhance individual strengths and remove obstacles. We interpret this in terms that Every student can have the chance to grow and learn. The UDL model is based on neuroscience, recognizing that every brain, with its unique neural networks, is distinctive like a fingerprint. What defines these neural networks are the experiences in which the individual is immersed. Hence, the importance of purposeful planning by the teacher. Teachers encounter challenges daily when planning for a diverse range of learners.

However, UDL Guidelines (Fig. 2) allow teachers to assist in designing inclusive activities.

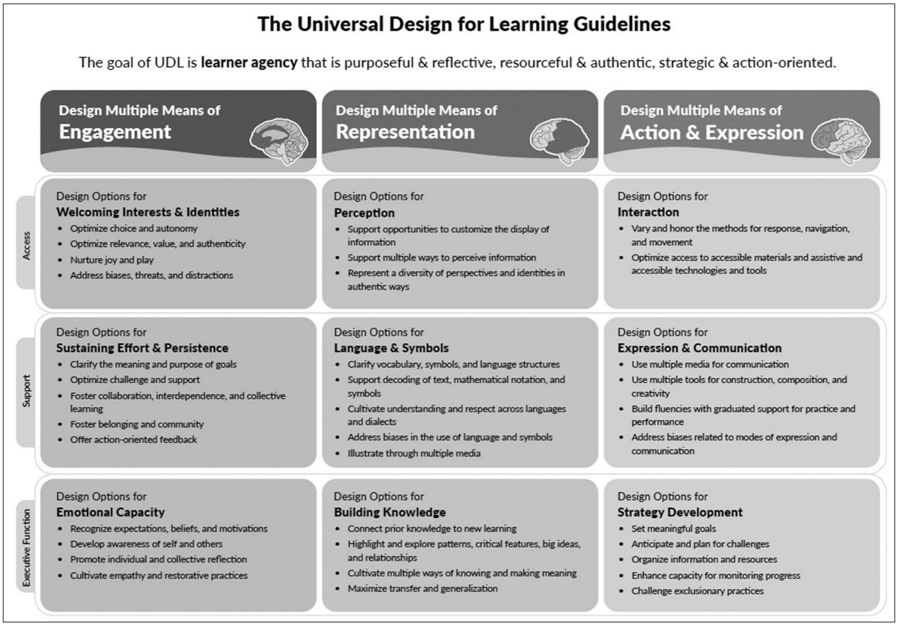

Fig. 2 – UDL Guidelines

Source: https://udlguidelines.cast.org/

The UDL Guidelines teachers to address learning diversity across three main categories: engagement (the why of learning): recruiting interest, sustaining effort and persistence, and self-regulation; representation (the what of learning): perception, language and symbols, and comprehension; action and expression (the how of learning): physical action, expression and communication, and executive function.

At the top of the graphic organizer, the primary goal of UDL is outlined: to cultivate a learner agency that is intentional and reflective, resourceful and authentic, as well as strategic and action-oriented. The UDL Guidelines are structured both vertically and horizontally.

Vertically, they are categorized according to the three core principles of UDL: engagement, representation, and action & expression. Each principle is further divided into specific Guidelines, which include detailed considerations to offer practical recommendations.

Horizontally, the Guidelines are arranged into three distinct rows. The *access* row focuses on increasing accessibility to learning goals by offering diverse options for fostering student interests and identities, enhancing perception, and facilitating interaction. The *support* row provides strategies to sustain learning by incorporating choices that promote effort and persistence, facilitate understanding through language and symbols, and encourage varied forms of expression and communication. Lastly, the executive function row highlights approach to strengthening learners' *executive functioning* skills through emotional regulation, knowledge construction, and strategic planning.

For more details, we refer you to the website related to the UDL Guidelines[4].

In the following, we present examples of how the community of inquiry with which the authors collaborate as researchers has applied the UDL model to design inclusive educational activities, using UDL both as a design framework and as a lens through which to interpret the intervention.

6. How to design with UDL: an example from a community of inquiry

The community of inquiry (Jaworski, 2006), named LabDidMath[5], comprises kindergarten and primary school teachers alongside researchers in Mathematics education. Its primary aim is to support the personal and pro-

[4] https://udlguidelines.cast.org/.

[5] https://sites.google.com/view/labddm/gruppi-di-ricerca-azione; https://sites.google.com/view/labddm/home.

fessional growth of its members and to develop tools that address emerging needs identified through reflective practice. Specifically, LabDidMath promotes the co-design of both theoretical and practical tools, such as inclusive educational pathways, for implementation within educational settings.

In the following, we will provide an example of a sequence of activities, carried out in third, fourth, and fifth-grade classes, which were progressively revised following a design-based research framework (Gravemeijer and Cobb, 2006; Design-Based Research Collective, 2003; Cobb *et al.*, 2003). The principles illustrated in the preschool example – necessity and multimodal engagement – are extended and operationalized in a fraction instructional sequence, demonstrating continuity and verticality in our inclusive approach from early childhood to primary school.

Observations, students' works, and teacher reflections were systematically collected to inform iterative refinements of the activities and to monitor the learning potential. Each activity exemplifies one or more UDL principles: engagement is fostered through problem-centered narratives, representation is enhanced via multiple registers (visual, symbolic, digital), and action & expression is supported through hands-on and interactive tools. This shows how UDL is used as a tool for designing educational activities, ensuring that the sequence is both inclusive and effective.

In order to monitor students' progress, observation, analysis of work products (e.g., fractional units, worksheets), the ability to communicate reasoning using multiple representations, and teacher reflections were used. This also demonstrates how UDL can serve not only as a design framework but also as a lens for interpreting and monitoring the teaching-learning process.

The fraction instructional sequence involved working with various artifacts, such as A4 sheets of paper, squared-paper strips, and their representations in notebooks.

The initial phase of the activities centered on using A4 sheets of paper. Students collaborated in groups for the 'Partitioning of the A4 sheet of paper' activity. They divided the A4 sheet, designated as the unit, into equal parts through folding and ruler measurements. What you get are fractional units of the A4 sheet (halves, thirds, quarters, etc.) in different but equal-area shapes (e.g., the "pieces of paper" representing 1/2 can have different shapes, but they are equal-area). The different fractional units can be used to recompose the A4 sheet (see Fig. 3). This process introduced the concepts of *equivalent fractions*, as equivalent areas, and the *sum of fractions* as a method to reconstruct the whole unit (A4 sheet).

The necessity principle, essential for the Problem-Centered Approach situation, which aligns with the first principle of UDL, arises from a narrative context: Mino, a pizza maker, asks students to create placemats for his restaurant. To explore the construction of a whole from fractional units, the problem requires students to cover a sheet of paper (representing the placemat) using various fractional units in different combinations (Fig. 3). This activity offers an option for welcoming interest, and it adopts a multi-modal approach, as children cut, touch, and manipulate materials to build understanding.

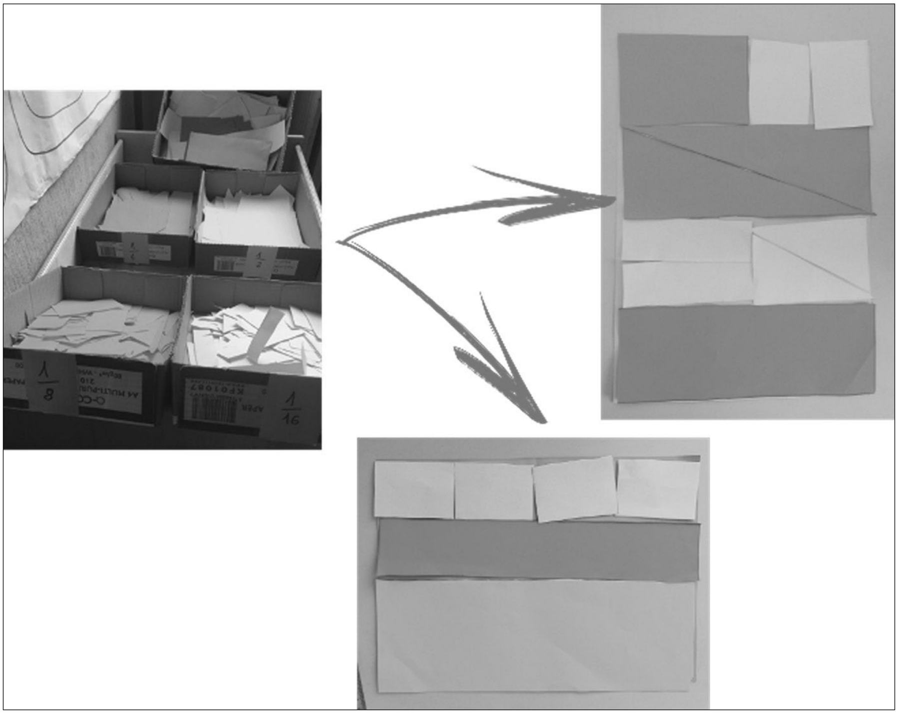

Fig. 3 – Two placemats (A4 sheet) composed of different fractional units

To illustrate the second principle, providing multiple representations for problem-solving, we can consider the case of Marco, a student struggling with fractions addition. He struggles to understand the necessity and process of calculating the lowest common denominator. Then he asked the teacher: "Why is it done this way? If I understand the reason, maybe I'll be able to remember how to do it".

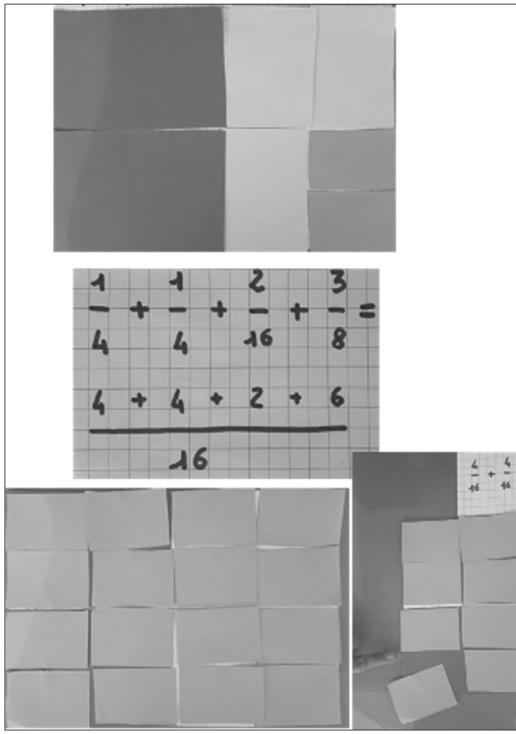

Fig. 4 – The solution proposed by the teacher to represent the LCM when adding fraction

At the top, the representation of 1/4 + 1/4 + 2/16 + 3/8 as fractional units in colored paper. At the bottom, the substitution of the quarters with 4 sixteenths in paper fractional units, and each third with 2 sixteenths in paper fractional units. The A4 sheet has been covered with 4 + 4 + 2 + 6 sixteenths

This situation underscores the importance of developing solutions that extend beyond symbolic representation. Marco requires a visual understanding of fraction addition. To facilitate this, fractions are represented as tangible paper fractional units, which are then subdivided into sixteenths and summed (as shown in Fig. 4). This approach is not intended to be Marco's permanent method to solve this type of task. Rather, this visual experience is designed to reinforce his ability to recall the rule for fraction addition. The sum of fractions can be effectively conveyed through various representational registers, encompassing both visual and symbolic forms (Fig. 5).

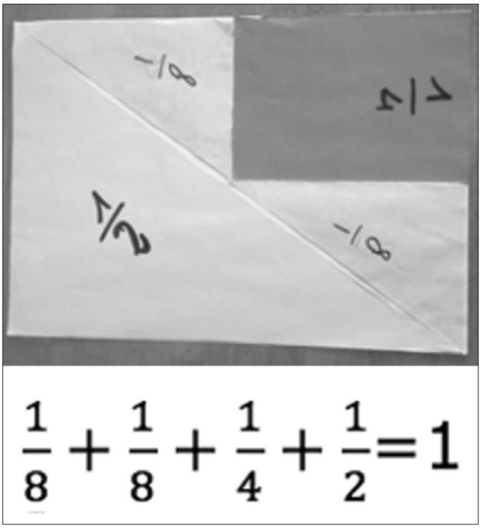

Fig. 5 – The sum of fractions is represented in both visual and symbolic registers

Additionally, equivalent fractions are illustrated through overlapping pieces of paper, emphasizing their equal area (Fig. 6).

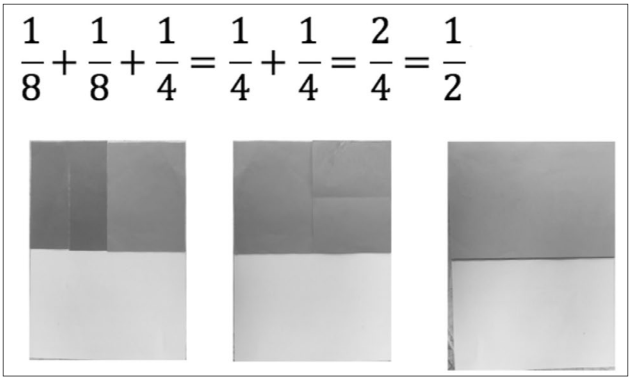

Fig. 6 – Equivalent fractions illustrated through overlapping pieces of paper

Two fractional units of 1/8 can be overlapped with the fractional unit of 1/4. Two fractional units of 1/4 can be overlapped with the fractional unit of 1/2

This illustrates how multiple representations are crucial for accessing the information needed to solve a problem. Turning to the third principle, which advocates for providing diverse means for students to act and communicate their solution strategies, we present examples utilizing both concrete tools

– such as ordering fractions on a number line represented by a string hanging between two adjacent walls in the classroom (Fig. 7) or demonstrating fraction equivalence with manipulatives (Fig. 6 and Fig. 7) – and digital tools (Fig. 8).

Fig. 7 – Teacher hanging paper cards with fractions between 0 and 1 on the string

String on the wall where the position of the fractional unit is defined based on the position of the whole numbers (0, 1, 2, ...) and is made to vary dynamically by sliding the corresponding label

The dynamic nature and immediate feedback offered by digital tools[6] encourage mathematical thinking through a multimodal approach. Simultaneously, offering diverse tools supports students' thinking process and their ability to articulate and share this thinking with others.

[6] https://www.geogebra.org/m/ffx8dhaj?fbclid=IwAR3O.

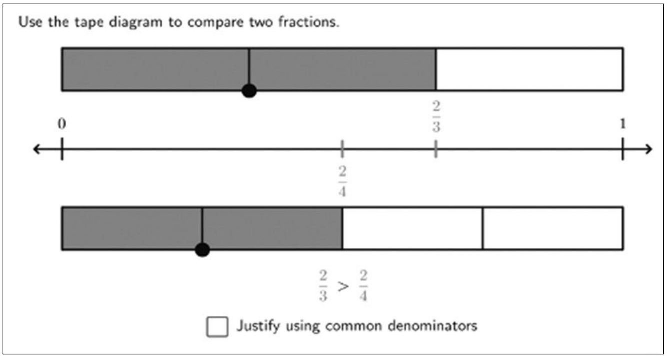

Fig. 8 – Comparing fractions with unlike denominators using visual models in an interactive activity

In this regard, Fig. 9 illustrates how a student represented the solution to the task "Ordering fractions along a number line" by utilizing tools developed in previous activities – specifically, paper fractional units created for the placemat. Notably, the student used various fractional units in different combinations, rather than relying solely on symbolic notation (as in Fig. 7). This shows how using various means of action and representation helps both to think and communicate one's own thinking.

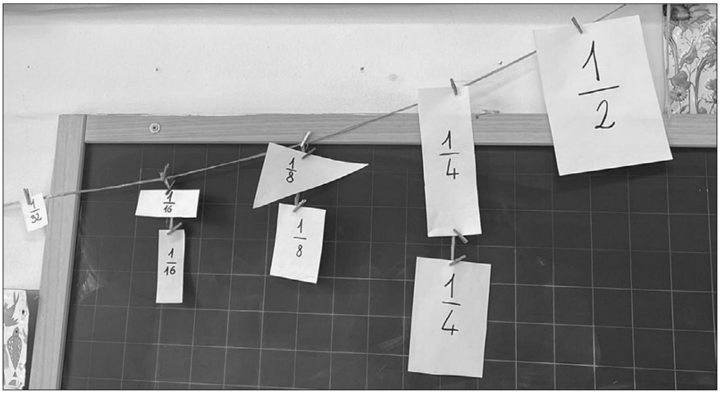

Fig. 9 – "Ordering fractions along a number line" task, using paper unit fractions

7. Conclusion

In our contribution, we claim that, according to relevant literature in education, even for Mathematics education to effectively address dropout and, particularly, implicit dropout, it is crucial to overcome fragile learning from the earliest years of education and across all contexts. This requires fostering an approach to learning that is both stable and meaningful, ensuring that every and each students have access to rich and significant mathematical experiences from the very beginning.

According to our vision of Mathematics education, starting from the early years of schooling, we primarily need to share the belief that the main goal we aim to achieve in school is to promote robust and long-lasting learning. Knowledge must be strong enough for students to apply it to new problem situations, reinforcing their confidence and competence over time. Secondly, we need to share a common view on how to do it, and we then refer to the accessibility (multimodality) and meaningfulness (problem-centred education) of the learning experiences for each and every student. Learning must be designed to be both universally accessible and personally significant. This means not only providing equitable opportunities but also ensuring that the knowledge acquired is relevant and engaging for each student.

For learning to be meaningful, the learning process has to be embedded in a context that makes sense to students. If, in the past, meaningful learning has often been supported by external experiences beyond the school setting, in an increasingly impoverished society, we can no longer assume that students will encounter enriching contexts that give purpose to their mathematical education outside of classrooms. Many students have become increasingly deprived of valuable opportunities outside the schools and, therefore, fewer learners can engage with real-life experiences that allow them to rely on the significance of what they learned in the classroom.

This phenomenon is primarily driven by social impoverishment, resulting in students having fewer stimulating experiences, in general. Therefore, this impact is particularly severe for lower-income social groups. However, this is also because, regardless of social background, the world is becoming increasingly less suited to children's needs. Consider, for instance, numerical competencies, number senses, and basic arithmetic operations. Until recently, these skills were naturally reinforced in everyday situations, such as when children needed to buy an ice cream or a newspaper. Often, families and the whole society play a role in fostering these foundational skills in such contexts. As a result, even if students had not fully grasped the concept of numerical value in school, they would often develop a robust understanding through real-world experiences outside the classroom, which in turn may support their formal Mathematics education. In today's world, however, credit cards have almost entirely replaced coins, depriving children of insightful opportunities to engage with these fundamental concepts tangibly.

To sum up, we can no longer assume that students will encounter stimulating environments outside school that give significance to their mathematical learning. As a result, it has become urgent for schools to take responsibility for providing meaningful learning experiences within the classroom. Without this shift, many students will be left without the necessary foundation to engage with Mathematics in a way that is relevant to their lives. Then, many learners, especially the ones with a disadvantaged socio-economic background, risk disengagement and eventual dropout.

This is not just about constructing knowledge or teaching methods, but about shaping the way students learn. A thoughtfully integrated teaching-learning approach is essential to developing students' thinking structures and patterns in a way that fosters deeper engagement, cognitive activation, and interest. Starting from primary school – and even earlier, in preschool – this approach ensures that Mathematics education is not only comprehensible but also meaningful. In this sense, inclusion is not simply a matter of equity in access; it is about guaranteeing that all students can find relevance and significance in their learning, preventing fragile learning from taking hold, and creating a strong foundation for lifelong mathematical engagement.

In our work, we addressed the problem of how to design inclusive educational activities. To this end, we drew inspiration from UNESCO's guidelines, employing the UDL model to design inclusive educational activities from kindergarten and primary school onwards. The presented examples support this goal.

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The seminar dedicated to INVALSI data represents a valuable opportunity to explore crucial issues in education. For example, over the course of its various editions, the mathematics sessions have provided a valuable opportunity to analyse results, recurring errors, and the cognitive processes involved in solving test items in detail, thereby fostering targeted didactic reflection.
This volume collects six contributions presented at the ninth edition of the Seminar "Data from and for the educational system: tools for research and teaching" (Rome, 17–19 October 2024). Each contribution addresses key challenges in contemporary mathematics education from different yet complementary perspectives: linguistic complexity, socio-economic inequalities, the impact of the pandemic, teacher education, and the risk of school dropout.
Overall, these contributions provide a comprehensive overview of the factors influencing the learning of mathematics, including language, context, motivation, teaching quality, and inclusion. The volume thus constitutes a useful framework for reconsidering mathematics as a formative experience that can foster participation, equity and meaning.

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