The Renewed Existence in AR of Max Brückner's Lost Paper Polyhedra

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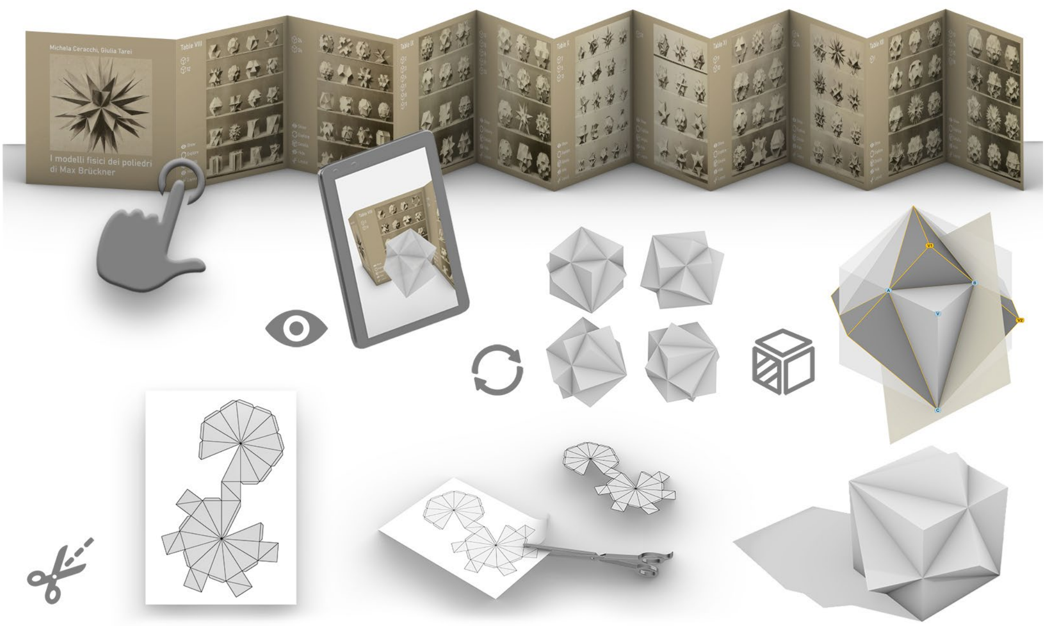
Abstract

This study proposes the realization of an Augmented Reality museum of mathematical-physical models for distinctly educational purposes and capable of explaining mathematical and geometric principles. Augmented Reality, in this context, would render them accessible while interacting with their digital twins would not endanger the preservation of the originals, and furthermore, comprehension would be enhanced thanks to the inclusion of explanatory contents. Following the teaching course proposed from our experience, it is possible to build the replica of one's paper mock-up, thus giving pedagogical value back to the mathematical models otherwise precluded when musealized in a showcase or, worse, safeguarded in storerooms due to limited exhibition space. The first trials concern Max Brückner's collection of polyhedra models, published in 1900 in *Vielecke und Vielflache: Theorie und Geschichte*, which offers the opportunity to evaluate the potentials of Augmented Reality to renew existence to a collection that is now lost.

Keywords
augmented reality, descriptive geometry, physical mathematical models, polyhedra, Brückner.



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Introduction

The present contribution [1] is part of the cultural backdrop of the rediscovery of mathematical-physical models in the context of teaching geometry and the valorization of a fragile cultural heritage, a characteristic due to the type of material used in the creation of these objects. A heritage that has long been almost forgotten due to a lack of interest that took place in a certain historical period [Apéry 2020] but which today is the focus of numerous studies for their valorization and dissemination, both in the physical and the digital world.

In this context, the present research proposes a new musealization of physical mathematical models based on the use of Augmented Reality with the scope, in particular, "to the possibility of recreating collections" [Maniello, Amoretti 2016], integrating an interactive educational experience that concludes with the construction of one's own paper models. This kind of project allows the exploration of Augmented Reality to overcome the limits of the physical model in their not being usable, typical of the traditional museum experience.

The collection chosen for the first experimentation (Fig. 1) is shown in the photographic plates published in 1900 by Max Brückner in *Vielecke und Vielflache: Theorie und Geschichte*, which the author himself defines as a compendium of consolidated knowledge up to that time of the theory of polygons and polyhedra, describing them from a theoretical and historical point of view [Brückner 1900, Preface].

In the Appendix to the treatise, the German mathematician includes seven lithographic Tables and five double photographic Tables with numbered paper polyhedra set and ordered on the shelves.

The importance of physical models on his work is likely due to the influence of Felix Klein, his advisor for the doctoral thesis and one of the principal proponents of the use of images and physical models in the development of mathematical intuition [Hart 2019].

The characteristics of the collection make it an important element to be inserted into the broader musealization project. In fact, it already assumes the aspect of a homogeneous collection but it is a *Wunderkammer* that elicits a nescient curiosity [Hart 2019].

Augmented Reality could add explanatory content to restore the educational value typical of mathematical physical objects. Furthermore, those displayed are paper models, easily replicated with common materials and tools, making them particularly suitable for an education experience tending to the construction of one's own model. But above all the actual collection no longer exists save for the photographic plates in the book [Hart 2019], so it would be possible to investigate the potentials for Augmented Reality to re-establish a rapport with a gone heritage, and furthermore, the replication in the creation of new paper models would bring it back to existence once more.

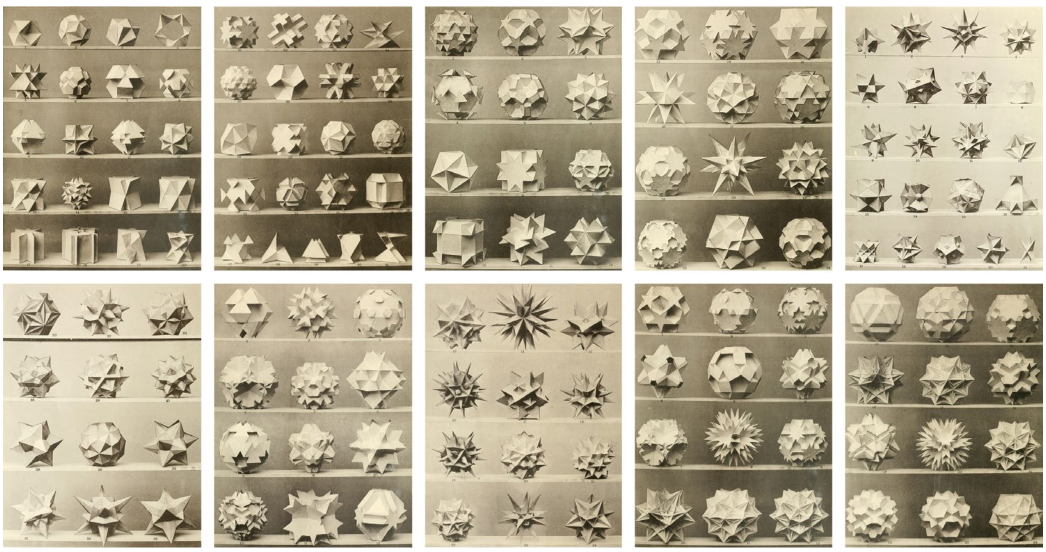

Fig. 1. The photographic plates published in *Vielecke und Vielflache* in 1900 by Joannes Max Brückner, show a selection of the collection of polyhedra created by the author.

The Musealization of Mathematical-Physical Models: Instruments and Prospects

The mathematical-physical models are objects conceived for distinctly educational purposes, representing mathematical laws and principles in space and capable of explaining the features of curves, algebraic surfaces, and complex polyhedra, revealing their spatial genesis. The greatest strength resides in their materializing an immaterial construct of ideas composed of abstract signs on a tangible object [Seidl, Loose, Bierende 2018, p. 20; Pavignano, Cumino, Zich 2020].
The physical models were the main protagonists of the *didactic theaters* [*teatri didattici* Gay 2000] that characterized the teaching of math and geometry between the end of the nineteenth and the first half of the twentieth century. The use of these models spread thanks also to the publication of numerous catalogs, aimed at promoting sales, in which the models were often presented employing text, graphic representation, and analytic description [Pavignano, Cumino, Zich 2020]. It is precisely thanks to these publications and the few surviving collections that the information regarding them has come down to us. Presently the models are preserved in the display cases of university workshops or museum showcases but more often in their storerooms, denying any utilization.
In recent years interest has been renewed regarding these collections, motivated by two main intents. The first has to do with the possibility these objects offer to render mathematics visible [Hart, Heathfield 2017], an aspect of interest for those studies that focus on the leading role of the geometric quality in the definition of the form in architecture [Pottmann, Asperl, Hofer, Kilian 2013; Valenti 2019; Baglioni, Salvatore, Valenti 2020; Salvatore, Baglioni, Valenti, Martinelli 2021]. The other has to do with the aesthetic significance of these objects, able to inspire artistic experimentation even now, as demonstrated by the works of the American artist and mathematician George Hart [Hart 2005; Hart 2005a]. The ability to make the genesis of very complex forms on a physical object clear had already fascinated the vanguard artists who investigated the generative processes, even replicating the formal outcomes [Baglioni, Farinella 2017].
The renewed interest in these collections has led to many experimentations aimed at their valorization. Some concentrate on the valorization of the physical consistency of the surviving collections, as is the case in the vast collection of models created by Alexander von Brill's students at the University of Tübingen [Seidl, Loose, Bierende 2018, p. 15], where others promote the integration of the collections via the modern technique of rapid prototyping, such as the case of the Henri Poincaré Institute in Paris [Apéry 2020]. Still others are directed towards the digitalization of their collections [2], which, however, lack the tactile experience by which the hand which grasps learns, the hand that grasps comprehends [Di Napoli 2004, p. 38].
In this context, Augmented Reality would allow the fruition of the digital model by better simulating the perception of a real object in 3D space; being a link between the real and the virtual, between physical and digital space, it can create the illusion of depth despite it being visualized on the screen of a device. The use of Augmented Reality also has a great impact on educational activity [Lin, Chen, Chang 2015], to improve, optimize, and render both the teaching and learning process more immediate. The success of the use of digital technologies has already been recorded in terms of interest and results in both teaching [Spallone, Palma 2021] and museum [Luigini, Panciroli 2018]. Furthermore, the potentialities of Augmented Reality in the scope of the communication of long gone objects or sites have been extensively investigated [Empler 2019] and results demonstrate how its use can restore their configuration as well as re-establish a rapport with them to a far greater extent than is possible with a virtual reconstruction.
Finally, to understand the three-dimensional configurations of geometric forms or problems of spatial nature, the construction of one's own physical model is the most suitable tool. Studies regarding teaching experiences that make use of the physical model for the understanding of problems of a geometrical nature have shown how this practice is still relevant and fundamental for students' development [Cumino, Pavignano, Zich 2020; Spadafora 2020; Cumino, Pavignano, Zich 2021].

The Collection Through Augmented Reality: an Interactive Pedagogical Experience

The main objective of this museum proposal is to restore life, using Augmented Reality, to a lost collection. To this end it is necessary to know the characteristics of the collection itself; all the models contained in it represent non-convex polyhedra, which, due to their complexity, are not immediately recognizable merely by observing the photographs [Mikloweit 2020]. The lack of organization of the collection [Hart 2019] and any element that helps make the represented polyhedra easily recognizable, render their classification even more arduous. The models are distributed on the shelves according to size, in order to save space [3] [Brückner 1900, p. 183]. In this way, however, the polyhedra belonging to the same category, such as the Kepler-Poinsot polyhedra, are randomly distributed among the plates. The author explains the absence of colour as well, generally used in mathematical models of polyhedra to provide characteristics and genesis of the represented form: the paper models were coated with homogeneous white plaster to avoid problems with the collotype reproduction of the photographs [Brückner 1900, p. 183].
George Hart has identified and classified some of the polyhedra of the collection precisely to show how it is full of interesting and also very complex examples, among which for example the American mathematician identified ten stellations of the icosahedron, many of which were constructed for the first time specifically by Brückner, including the complete stellation [Hart 2019].
The classification was carried out by means of different approaches. Firstly, an analysis of the models illustrated has permitted the identification of some of the best-known polyhedra, some prisms, and antiprisms. With the help of some texts on polyhedra [AA.VV. 1938; Coxeter 1948; Wenninger 1971; Wenninger 1983], it has been possible to classify many other polyhedra, by comparing the models in the collection with those illustrated in some of these publications.
To systematize the contents of a collection of this kind means making it understandable and complete, so as to allow their promulgation. Organizing the collection by categories of affine polyhedra and adding explanatory contents makes the definition of the educational path possible that would restore to the collection its pedagogical value. Furthermore, classifying the polyhedra makes their characteristics and geometric genesis known and therefore makes their graphical, digital, and physical construction possible.
The integration of an interactive learning experience introduces an important issue to reflect upon, that is the pedagogical problem of a balance between traditional and digital instruments [Spadafora 2020]. In fact, this experience was conceived as a series of various activities. The common thread is precisely Augmented Reality.

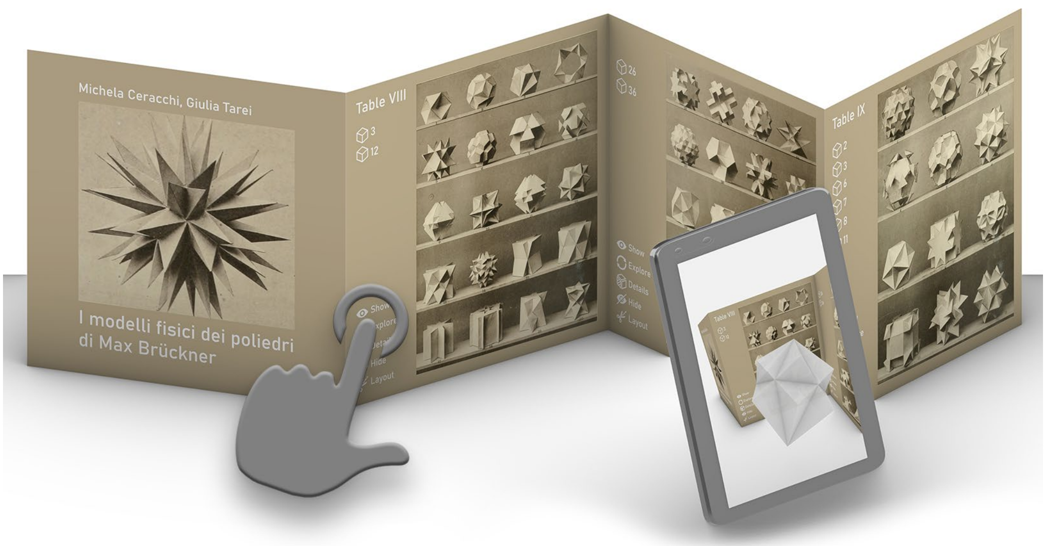

Fig. 2. The photographic plates illustrating the collection become the target by which the Augmented Reality experience is triggered.

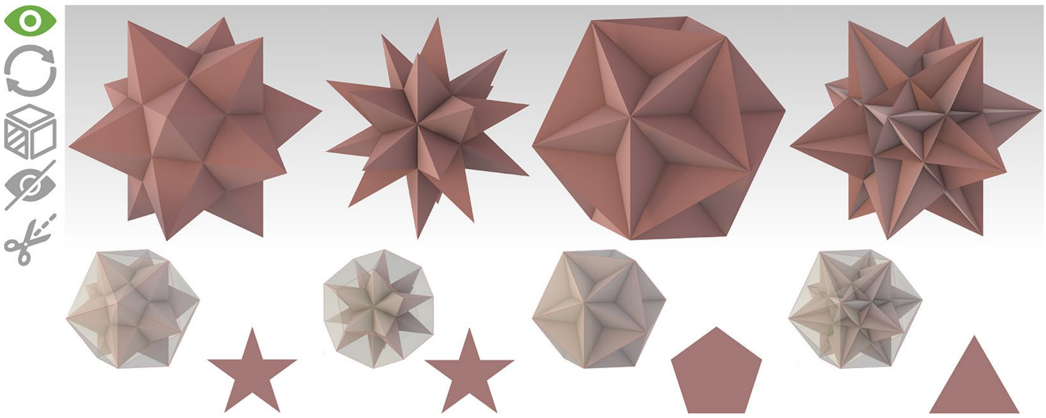

Fig. 3. The first proposed activity: visualization in Augmented Reality of the digital models of the polyhedra.

In this proposal, the tables of the treatise become the means by which the experience in Augmented Reality is triggered (Fig. 2). In that respect, it is necessary to keep in mind the substantial difference that exists between Virtual and Augmented Reality: virtual is synonymous with illusory [Maldonado 2015], whereas, in Augmented Reality, the relationship between the real world and digital content can be consolidated through the target [Bianchini, Fasolo, Camagni 2020].
The activation of the digital content in Augmented Reality permits, first of all, the visualization of the digital model to comprehend the spatial configuration of the polyhedra shown and compare the various models to deduce the relationship. The possibility of visualizing the models together that have been classified in the same category is useful in understanding the relationship between them. For example, this comparison in the case of Kepler-Poinsot polyhedra is useful one can understand the principle of duality that exists between the first two, the small and great stellated dodecahedron, and the following two, the great dodecahedron and the great icosahedron (Fig. 3).
The AR exploration of the digital model to be observed and redrawn (Fig. 4), assimilates it to the plaster models to be copied in drawing exercises, which consolidates the "aptitude in sequential reasoning favoured also by the use of the hands" [Spadafora 2020] because it participates in that rapport mind-eye-hand that if exercised stimulates graphic intelligence, which is the capacity to solve spatial problems through observing, understanding, and representing, as did Galileo regarding the lunar surface [Cicalò 2016], explorable only through observation, exactly as with this collection which no longer exists in its physical form. Orseolo Fasolo, in a letter to Riccardo Migliari, dated August 14, 1987, writes of the need to once more place the physical model at the heart of teaching Drawing because copying a model one learns to draw, especially "if the model is a geometric shape, one learns geometry" [Migliari 2001, p. 277]. These words are still very relevant because drawing is a fundamental passage in the comprehension of a tridimensional shape even when the aim

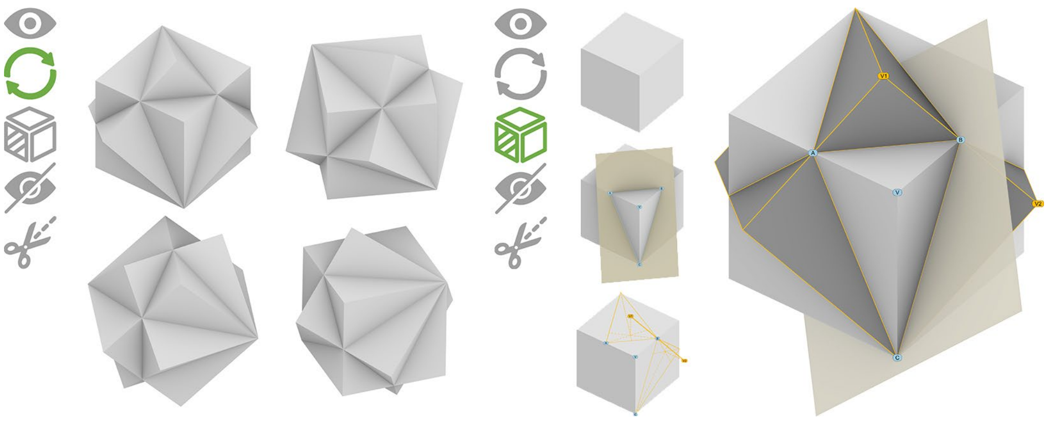

Fig. 4. The second proposed activity: Augmented Reality exploration of the polyhedron, using it as a model to be copied in the drawing to comprehend its configuration (left); the third proposed activity: the study of the characteristics and genesis of the polyhedron in question through the explanatory contents that can be viewed in Augmented Reality on the digital model (right).

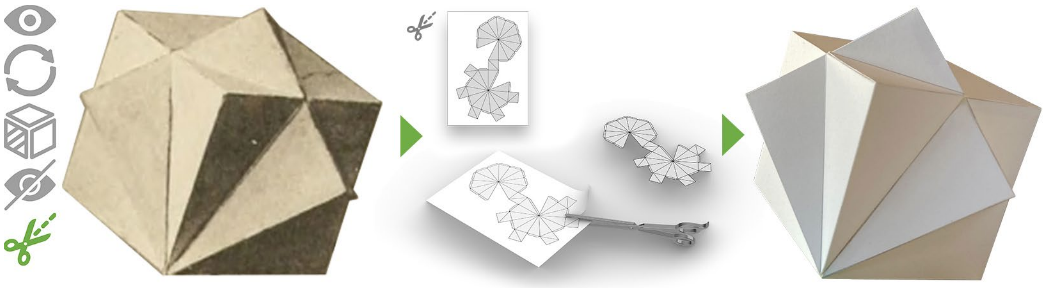

Fig. 5. The last activity: creating one's own paper model starting from the printable planar development, provided as additional material.

is digital or physical modelling so the knowledge and application of Descriptive Geometry are more important than any technological development [Rochman 2020].
The addition of explanatory material integrated with the digital model (Fig. 4), on the geometrical characteristics of the polyhedron and its creation, permits the acquisition of the necessary notions for an autonomous reconstruction of the digital model. For example, in even the simple case of the composition of two cubes, it is possible to investigate the properties of the resulting polyhedron and understand the stages of construction. In this manner, the experience emerges as an auxiliary aid in teaching the theory of polyhedra.
Furthermore, the integration of explanatory material to the digital model convey to these models the distinctive features of the mathematical model to clearly render the geometric characteristics that the form represents, thus far missing for the reasons given by the author himself [Brückner 1900, p. 183], and re-endowing to Brückner's collection that pedagogical value not entirely expressed in these photographic tables [Hart 2019].
Lastly, one's own construction of a paper model starting from the printable planar development, offered as supplementary material (Fig. 5), concludes this itinerary of knowledge of the polyhedra in the collection. The possibility of constructing by hand one's own model has important applicational relevance in the teaching of geometry and of representation because it exploits the educational worth of craftsmanship and manual work [Albers 1944]. Magnus Wenninger, in the introduction to *Polyhedron Models*, invites the reader to build their own paper model to gain a formal proof of the reasoning underlying the creation of polyhedra [Wenninger 1971], but even recent proposals in teaching context encourage the creation,

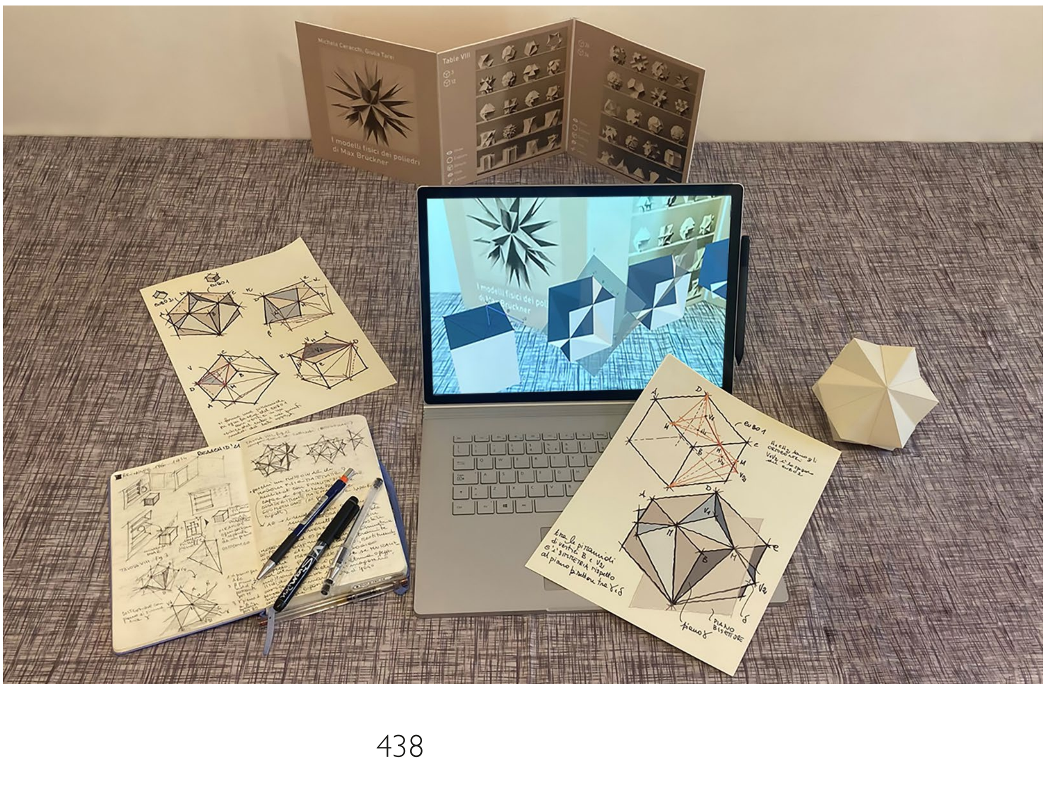

Fig. 6. Synthesis of the interactive educational experience.

on their own, of tangible objects to understand problems of a geometric nature [Cumino, Pavignano, Zich 2020 ; Spadafora 2020; Cumino, Pavignano, Zich 2021].
At the same time, each paper polyhedron created through this experience will contribute to giving renewed existence to the lost collection.

Conclusion

The originality of this project of musealization and interactive didactic activity consists precisely in the integration of the experiences of a diverse nature in a unitary project. The aim of this proposal is the revival of those *didactic thaters* in which physical models were the protagonists [Gay 2000], modernizing them in communication with the new generations thanks to the continuous interchange with the digital model in Augmented Reality.
The experimentation of this didactic activity and the subsequent validation through consolidated models of learning evaluation would allow to demonstrate its potential in the educational scope.
The broader project of musealization of mathematical physical models could be configured as an autonomous virtual museum, or as an experience to be inserted in a real museum context such as, for example, MoMath in New York – of which Hart himself was a co-founder – where the exhibits and interactive experience were designed to stimulate curiosity and evoke wonder. Linking the collections that can be explored in Augmented Reality to the real exhibits, inclusivity and accessibility can be enhanced – even concerning the recent global situation following the pandemic – and to the true museum experience in museums, it adds an interactive experience between the digital and the real world.
In this manner, Augmented Reality becomes the vehicle for knowledge and, in the specific case of Brückner's collection, the instrument through which the physical models can re-emerge from the two-dimensional condition of the photographic plates in which they are immortalized, bequeathing this cultural heritage now gone.

Notes

[1] The research was carried out jointly by both authors, who, together, wrote the sections "Introduction", "The Musealization of Physical Mathematical Models: instruments and prospects", and the "Conclusion". The analysis in the treatise was carried out by Giulia Tarei. The classification and musealization project in AR were carried out by Michela Ceracchi, who wrote the section "The Collection in Augmented Reality: an interactive educational experience" and created all the images.

[2] By way of example, we note that the digital collection of the University of Rostock, Germany, makes explorable digital models of some mathematical physical models from their collection.

[3] The models, as the author notes [Brückner 1900, Preface], were photographed by the firm Römmler & Jonas di Dresda and printed in the tables by collotype. The author furthermore invites the reader to freely visit the collection he owns.

References

Albers Josef (1944). The Educational Value of Manual Work and Handicraft in Relation to Architecture. In Zucker Paul (ed.). *New Architecture and City Planning: A Symposium*. New York: Philosophical Library, pp. 688-694.

Apéry François (2020). The Cabinet de mathématiques of the Henri Poincaré Institute in Paris. In *Kwartalnik Historii Nauki I Techniki*, 65 (3), 2020, pp. 97-108.

Baglioni Leonardo, Farinella Cristian (2017). Modello come immagine mentale. La diffusione dei modelli matematici nell'arte d'avanguardia e la superficie sviluppabile di Antoine Pevsner. In Di Luggo Antonella; Giordano Paolo, Florio Riccardo, Papa Lia Maria, Rossi Adriana, Zerlenga Ornella, Barba Salvatore, Campi Massimiliano, Cirafici Alessandra (eds.). *Territori e frontiere della Rappresentazione*. Firenze: Gangemi, pp. 929-936.

Baglioni Leonardo, Salvatore Marta, Valenti Graziano Mario (2020). Verso una musealizzazione della forma. In Arena Adriana, Arena Marinella, Brandolino Rosario Giovanni, Colistra Daniele, Ginex Gaetano, Mediati Domenico, Nucifora Sebastiano, Raffa Paola (eds.). *Connettere. Un disegno per annodare e tessere*. Milano: FrancoAngeli, pp. 47-66.

Bianchini Carlo, Fasolo Marco, Camagni Flavia (2020). Dimensioni, scale e rappresentazioni: un'eterna ghirlanda brillante. Dal 3D al 2D e ritorno. In *Paesaggio Urbano*, 2, 2020, pp. 149-157.

Brückner Max (1900). *Vieleche und Vielflache. Theorie und Geshichte*. Leipzig: Druck und Verlag von B.G. Teubner.

Cicalò Enrico (2016). Intelligenza grafica. In *XY digitale*, 1 (2), 2016, pp. 54-67.

Coxeter Harold Scott MacDonald (1948). *Regular Polytopes*. London: Methuen & co. Ltd.

Coxeter Harold Scott MacDonald, du Val Patrick, Flather H. T., Petrie John Flinders (1938). *The fifty-nine Icosahedra*. Toronto: The University of Toronto Press.

Cumino Caterina, Pavignano Martino, Spreafico Maria Luisa, Zich Ursula (2021). Geometry to Build Models, Models to Visualize Geometry. In *Digital Experiences in Mathematics Education*, 7 (1), 2021, pp.149-166.

Cumino Caterina, Pavignano Martino, Zich Ursula (2020). Paper models of architectural surfaces. In *IMG journal*, 2 (2), 2020, pp. 86-115.

Cumino Caterina, Pavignano Martino, Zich Ursula (2021). Proposta di un catalogo visuale di modelli per lo studio della forma architettonica tra Matematica e Disegno. In Arena Adriana, Arena Marinella, Mediati Domenico, Raffa Paola (eds.). *Connettere. Un disegno per annodare e tessere*. Milano: FrancoAngeli, pp. 626-645.

Di Napoli Giuseppe (2004). *Disegnare e conoscere. La mano l'occhio, il segno*. Torino: Einaudi.

Empler Tommaso (2019). *ITC per il Cultural Heritage. Rappresentare, comunicare, divulgare*. Roma: DEI, Tipografia del Genio Civile.

Gay Fabrizio (2000). *Intorno agli omolografi. Strumenti e modelli per la geometria descrittiva*. Venezia: Istituto universitario di architettura di Venezia.

Hart George William (2005). Creating a Mathematical Museum on Your Desk. In *The Mathematical Intelligencer*, 27 (4), 2005, pp. 14-17.

Hart George William (2005a). The Geometric Aesthetic. In Emmer Michele (ed.). *The Visual Mind II*. Cambridge: The MIT Press, pp. 215-234.

Hart George William (2019). Max Brückner's Wunderkammer of Paper Polyhedra. In Goldstine Susan, McKenna Douglas, Fenyvesi Kristóf (eds.). *Proceedings of Bridges 2019: Mathematics, Art, Music, Architecture, Education, Culture*. Phoenix: Tessellations Publishing, pp. 59-66.

Hart George William, Heathfield Elisabeth (2017). Making Math Visible. In Swart David, Séquin Carlo, Fenyvesi Kristóf (eds.). *Proceedings of Bridges 2017: Mathematics, Art, Music, Architecture, Education, Culture*. Phoenix: Tessellations Publishing, pp. 63-70.

Lin Hao-Chiang Koong, Chen Mei-Chi, Chang Chih-Kai (2015). Assessing the effectiveness of learning solid geometry by using an augmented reality-assisted. In *Interactive Learning Environments*, 23 (6), 2015, pp. 799-810.

Luigini Alessandro, Panciroli Chiara (2018). *Ambienti digitali per l'educazione all'arte e al patrimonio*. Milano: FrancoAngeli.

Maldonado Tomás (2015). *Reale e virtuale*. Milano: Feltrinelli.

Maniello Donato, Amoretti Valeria (2016). Paesaggi digitali in realtà aumentata per una nuova museologia. In *Romarchè, VII Salone dell'Editoria Archeologica*, 5 (2), 2016, pp. 129-142.

Migliari Riccardo (2001). L'insegnamento della Geometria Descrittiva e delle sue applicazioni. In Franchetti Pardo Vittorio (ed.). *La facoltà di architettura dell'Università di Roma "La Sapienza" dalle origini al duemila. Discipline, docenti, studenti*. Firenze: Gangemi, pp. 277-288.

Mikloweit Ulrich (2020). Exploring Noble Polyhedra with the Program Stella 4D. In Yackel Carolyn, Bosch Robert, Torrence Eve, Fenyvesi Kristóf (eds.). *Proceedings of Bridges 2020: Mathematics, Art, Music, Architecture, Education, Culture*. Phoenix: Tessellations Publishing, pp. 257-264.

Pavignano Martino, Cumino Caterina, Zich Ursula (2020). Catalog Mathematischer Modelle. Connessioni tra testo, rappresentazione grafica e descrizione analitica. In Arena Adriana, Arena Marinella, Mediati Domenico, Raffa Paola (eds.). *Connettere. Un disegno per annodare e tessere*. Milano: FrancoAngeli, pp. 3660-3677.

Pottmann Helmut, Asperl Andreas, Hofer Michael, Kilian Axel (2013). Architectural Geometry. London: Bentley Institute Press.
Rochman Dina (2020). The Language of Geometry. The Icosahedron as a Case Study. In Cheng Liang-Yee (ed.). *ICGG 2020. Proceedings of the 19th International Conference on Geometry and Graphics*. Berlin: Springer, pp. 211-221.

Salvatore Marta, Baglioni Leonardo, Valenti Graziano Mario, Martinelli Alessandro (2021). Forms in Space. AR Experiences for Geometries of Architectural Form. In Giordano Andrea, Russo Michele, Spallone Roberta (eds.). *Representation Challenges. Augmented Reality and Artificial Intelligence in Cultural Heritage and Innovative Design Domain*. Milano: FrancoAngeli, pp. 415-419.

Seidl Ernst, Loose Frank, Bierende Edgar (2018). *Mathematik mit Modellen*. Tübingen: Schriften des Museums der Universität Tübingen MUT, hrsg. von Bernd Engler und Ernst Seidl.

Spadafora Giovanna (2020). Modelli reali e virtuali per l'insegnamento della geometria descrittiva. In *Annali online della Didattica e della Formazione Docente*, 12 (20), 2020, pp. 125-142.

Spallone Roberta, Palma Valerio (2021). AR&AI in the Didactics of the Representation Disciplines. In Giordano Andrea, Russo Michele, Spallone Roberta (eds.). *Representation Challenges. Augmented Reality and Artificial Intelligence in Cultural Heritage and Innovative Design Domain*. Milano: FrancoAngeli, pp. 421-425.

Valenti Graziano Mario (2019). Disegni inutili per utili disegni. In Belardi Paolo (ed.). *Riflessioni. L'arte del disegno/il disegno dell'arte*. Firenze: Gangemi, pp. 211-216.

Wenninger Magnus Joseph (1971). *Polyhedron Models*. Cambridge: Cambridge University Press.

Wenninger Magnus Joseph (1983). *Dual Models*. Cambridge: Cambridge University Press.

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